On the Maximal Number of Real Embeddings of Spatial Minimally Rigid Graphs

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Rigidity in $\mathbb{R}^3$

An embedding $\rho\colon V \to \mathbb{R}^D$ of a graph $G = (V, E)$ is *compatible with edge lengths* $(d_{ij})_{ij \in E}$ if

$$\|\rho(i) - \rho(j)\| = d_{ij} \quad \text{for all } ij \in E\,.$$

Definition

A graph is *generically rigid* if the number of embeddings compatible with edge lengths induced by a generic embedding is finite modulo rotations and translations.

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- If G is rigid and $G - \{e\}$ is not rigid $\forall e \in E$, then G is *minimally rigid*.
- $D = 2$: Laman graphs (Capco, Gallet, Grasegger, Koutschan, Lubbes, Schicho, 2018)
- $D = 3$: Geiringer graphs

Algebraic Equations

Fix coordinates of a triangle to remove rigid motions.

$$(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = d_{ij}^2 \text{ for } ij \in E$$

- # complex solutions is an upper bound
- Loose mixed volume bound

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Sphere Equations

$$x_i^2 + y_i^2 + z_i^2 = s_i \text{ for } i \in V$$
$$s_i + s_j - 2(x_i x_j + y_i y_j + z_i z_j) = d_{ij}^2 \text{ for } ij \in E$$

Cayley-Menger matrix

$$CM = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & d_{12}^2 & \cdots & d_{1n}^2 \\ 1 & d_{12}^2 & 0 & \ddots & \dots \\ \dots & \dots & \ddots & \ddots & \dots \\ 1 & d_{1n}^2 & d_{2n}^2 & \cdots & 0 \end{pmatrix}$$

Theorem (Cayley, Menger)

The distances of a CM matrix are embeddable in $\mathbb{R}^D$ iff

- $rank(CM) = D + 2$, *and*
- $(-1)^k \det(CM') \geq 0$, *for every submatrix* CM' *with size* $k + 1 \leq D + 2$ *that includes the first row and column.*

Cayley-Menger matrix

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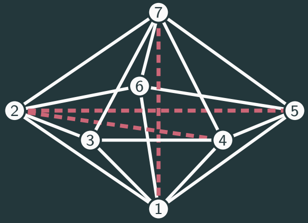
Theorem (Cayley, Menger)

The distances of a CM matrix are embeddable in $\mathbb{R}^3$ iff

- *$rank(CM) = 5$, and*
- *positivity, triangular and tetrangular inequalities must be satisfied.*

Distance geometry subsystems – Example

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 & d_{15}^2 & d_{16}^2 & x_1 \\ 1 & d_{21}^2 & 0 & d_{23}^2 & x_2 & x_3 & d_{26}^2 & d_{27}^2 \\ 1 & d_{31}^2 & d_{32}^2 & 0 & d_{34}^2 & x_4 & x_5 & d_{37}^2 \\ 1 & d_{41}^2 & x_2 & d_{43}^2 & 0 & d_{45}^2 & x_6 & d_{47}^2 \\ 1 & d_{51}^2 & x_3 & x_4 & d_{54}^2 & 0 & d_{56}^2 & d_{57}^2 \\ 1 & d_{61}^2 & d_{62}^2 & x_5 & x_6 & d_{65}^2 & 0 & d_{67}^2 \\ 1 & x_1 & d_{72}^2 & d_{73}^2 & d_{74}^2 & d_{75}^2 & d_{76}^2 & 0 \end{pmatrix}$$



- 21 equations in 6 variables
- Every solution of 3*x*3 subsystem corresponds to a unique embedding
- Eliminate two more variables using resultants

Algebraic System Solving

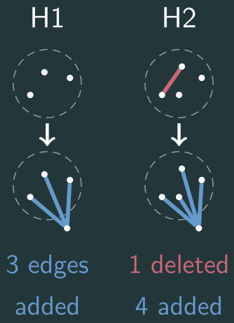
Homotopy Continuation

- PHCpack (Verschelde, 2014)
- Starting system based on structure of equations

`RootFinding` **package** (Maple 18)

- `Isolate` (Rouillier, 1999, Rouillier, Zimmermann, 2003, Aubry, Lazard, Moreno Maza, 1999, Xia, Yang, 2002)
- Algebraic sets over $\mathbb{R}$

Construction of Geiringer graphs



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- H1 steps double the number of embeddings

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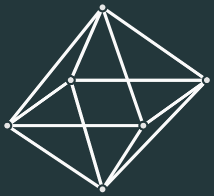
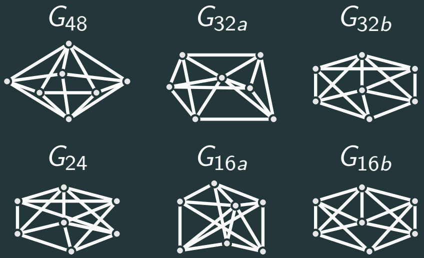


- H1 and H2 steps always rigid (sufficient for $|V| \leq 12$)
- H1 steps double the number of embeddings
- Known number of real embeddings for $|V| \leq 6$ (Emiris, Mourrain, 1999)

Geiringer graphs with 7 vertices

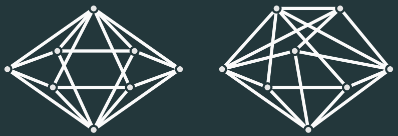
The graphs that cannot be constructed by H1 in the last step:



	G_{48}	G_{32a}	G_{32b}	G_{24}	G_{16a}	G_{16b}
MV of sphere eq.	48	32	32	32	32	32
MV of dist. subsyst.	48	32	32	24	24	16
# complex emb.	48	32	32	24	16	16

Geiringer graphs with 8 vertices

G_{128} G_{160}



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Maximizing the number of real embeddings

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- Local methods
  - Stochastic (7-vertex Laman graph – Emiris, Moroz, 2011)
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- Global methods
  - Huge size of parameter space
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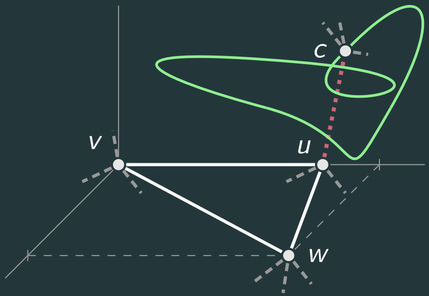
  - `RootFinding[Parametric]`
    (Liang, Gerhard, Jeffrey, Moroz, 2009)
- Global search over subset of parameters
  - Coupler curves (6-vertex Laman graph – Borcea, Streinu, 2004)

Coupler Curves

Removing an edge *uc* from a Geiringer graph breaks rigidity.

The curve traced by the vertex *c* is called a *coupler curve*.



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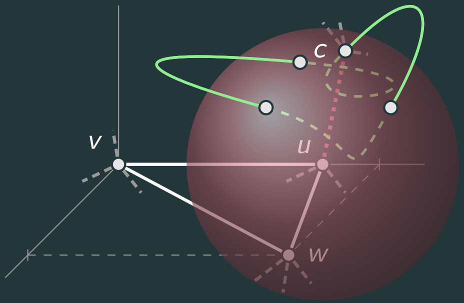
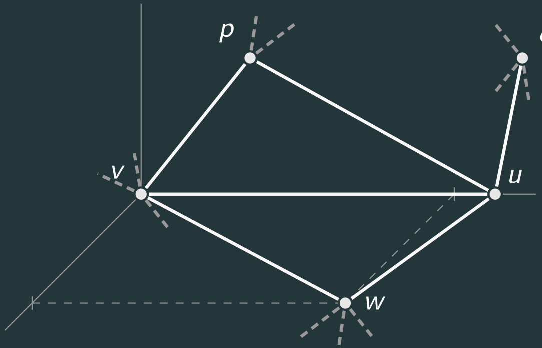


The real embeddings of G correspond to the intersections of the coupler curve with the sphere centered at u with radius d_{uc}.

Invariance of coupler curve



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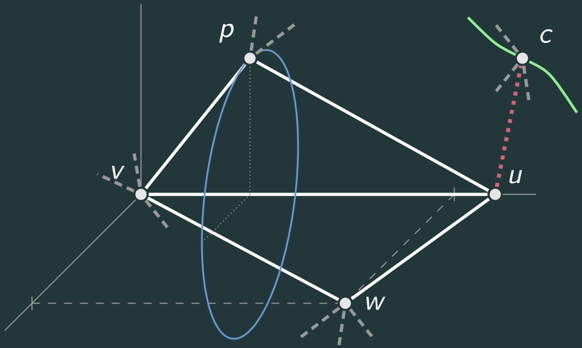
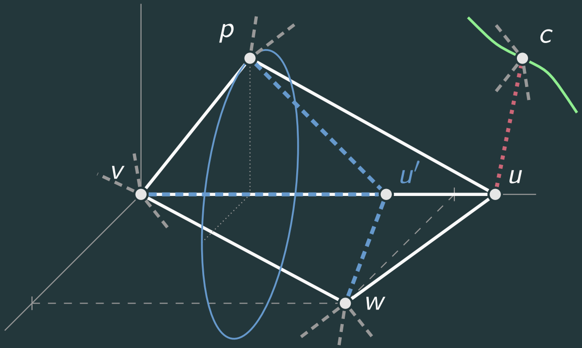
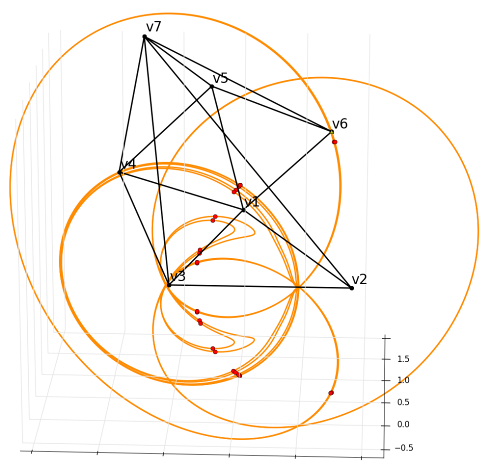


Invariance of coupler curve

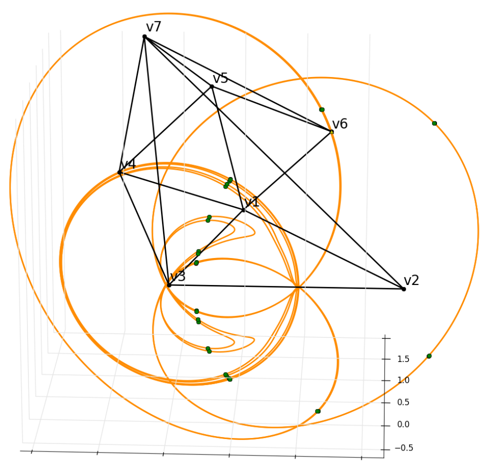


- The coupler curve of c is invariant to the position of u
- 2-parameter family changing 4 edge lengths
- Increasing the number of real embeddings

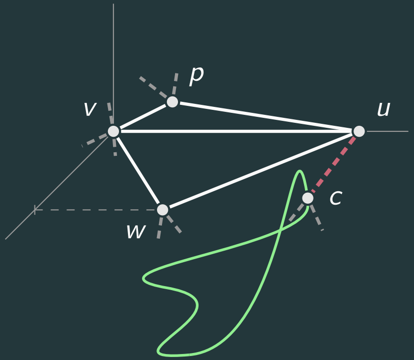
Example



Example

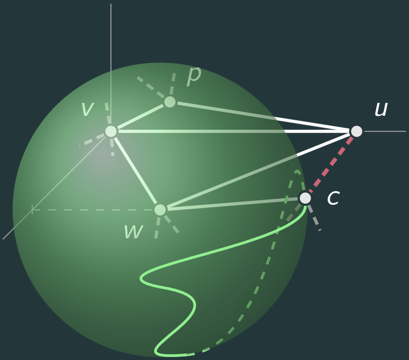


Sampling



Sampling

If *wc* is an edge, then the coupler curve of *c* is a spherical curve.



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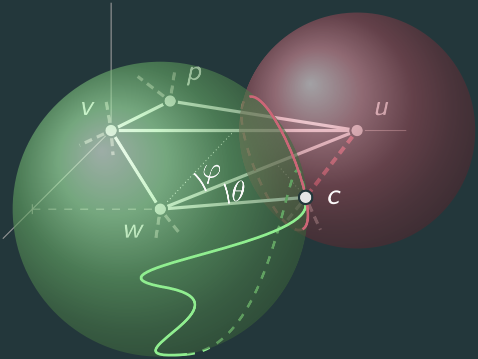
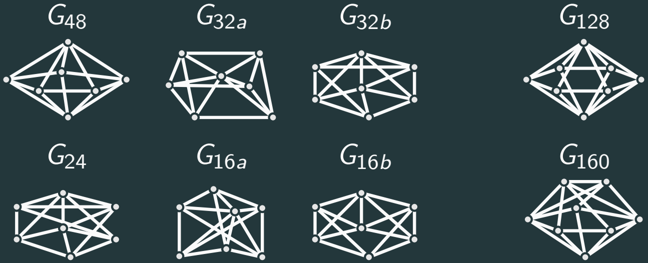


- Find φ and θ that maximize the number of real solutions
- Repeat the procedure with another suitable subgraph

Results



	G_{48}	G_{32a}	G_{32b}	G_{24}	G_{16a}	G_{16b}	G_{128}	G_{160}
# compl.	48	32	32	24	16	16	128	160
# real	48	32	32	24	16	16	128	≥ 132

Source code & results: jan.legersky.cz/project/spatialgraphembeddings/

Asymptotic bounds

Let n be the number of vertices.

- The number of real embeddings is at most

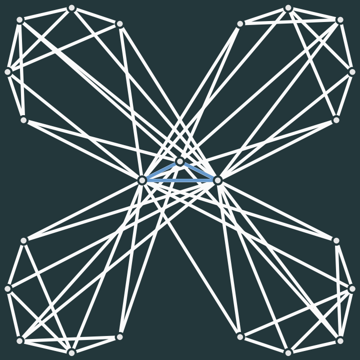
$$\frac{2^{n-3}}{n-2}\binom{3n-6}{n-3} \qquad \text{(Borcea, Streinu, 2004)}$$

  which behaves asymptotically as 8^n
- There are graphs with $\lfloor 2.51984^n \rfloor$ real embeddings (Emiris, Tsigaridas, Varvitsiotis, 2013)
- There are graphs with $\lfloor 3.0682^n \rfloor$ complex embeddings (Grasegger, Koutschan, Tsigaridas, 2018)

Lower bound on the maximum number of real embeddings

Theorem

There are graphs with $\lfloor 2.6553^n \rfloor$ *real embeddings.*



There are at least 132^k real embeddings, where k is the number of copies of G_{160}. For a graph with n vertices, $k = \left\lfloor \frac{n-3}{5} \right\rfloor$.

Future work

- Tight bound for the number of real embeddings of G_{160}
- Number of real embeddings of 9-vertex Geiringer graphs
- Improving lower bounds
- Other dimensions

Thank you

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