Flexibility of Penrose frameworks

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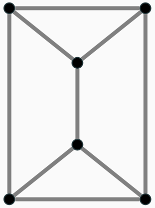
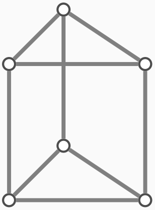
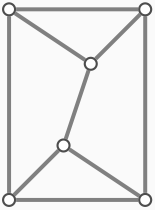


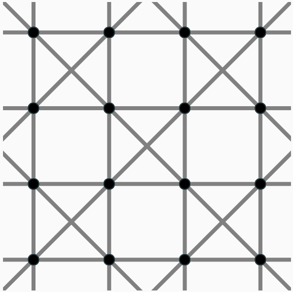
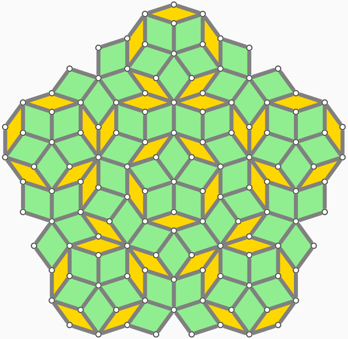


flexible



rigid

Graph $+$ realization in $\mathbb{R}^2$ $=$ framework

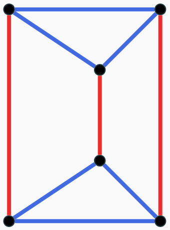
 

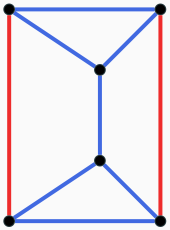
When does an infinite graph admit a flexible framework?

Is the given (piece of) Penrose framework flexible?

Definition

A coloring of edges $\delta : E_G \to \{\text{blue, red}\}$ of graph $G = (V_G, E_G)$ is called a *NAC-coloring* if it is surjective and for every cycle in G, either all edges in the cycle have the same color, or there are at least two blue and two red edges in the cycle.





Theorem (Grasegger, L., Schicho, 2019)

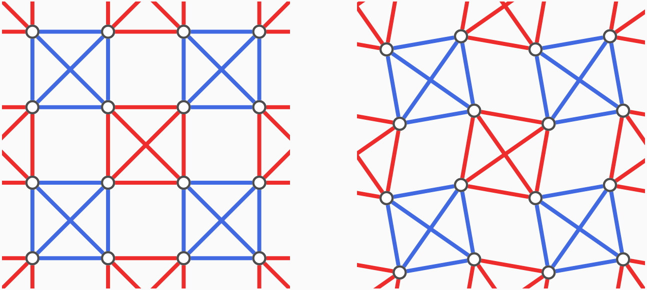
A connected finite graph admits a flexible framework in the plane if and only if it has a NAC-coloring.

Theorem (Garamvölgyi, 2022)

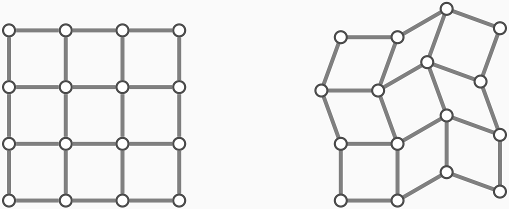
The question whether a NAC-coloring exists for a given graph is NP-complete.

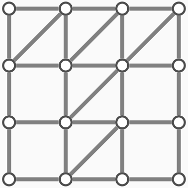
Theorem (Dewar, L., 2021+)

A connected countably infinite graph admits a flexible framework in the plane if and only if it has a NAC-coloring.

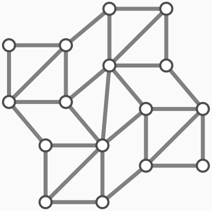


Frameworks consisting of parallelograms





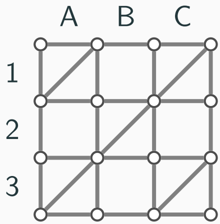
rigid

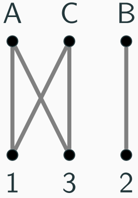


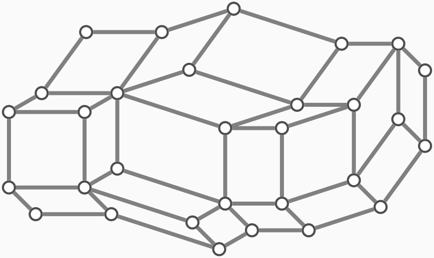
flexible

Theorem (Bolker, Crapo, 1979)

A finite braced grid of squares is infinitesimally rigid if and only if the corresponding bracing graph is connected.

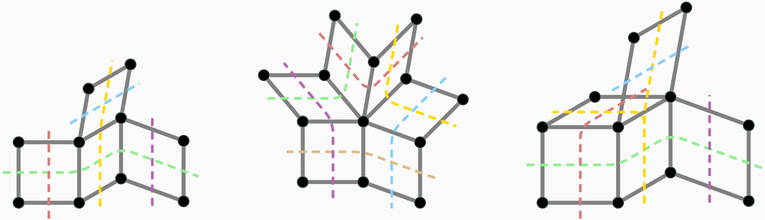






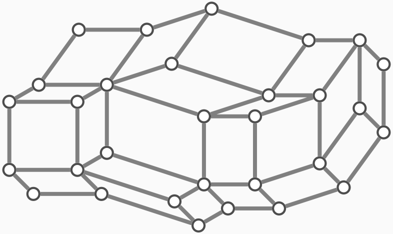
Definition

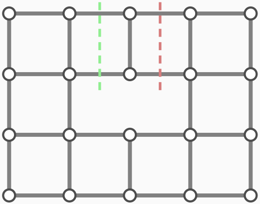
Consider the relation on the set of edges, where two edges are in relation if they are opposite edges of a 4-cycle subgraph of G. An equivalence class of the reflexive-transitive closure of the relation is called a *ribbon*.



Definition

A graph G is called *ribbon-cutting graph* if it is connected and every ribbon is an edge cut. If $\rho : V_G \to \mathbb{R}^2$ is an injective map such that each 4-cycle in G forms a parallelogram in ρ, we call the framework (G, ρ) a *P-framework*.



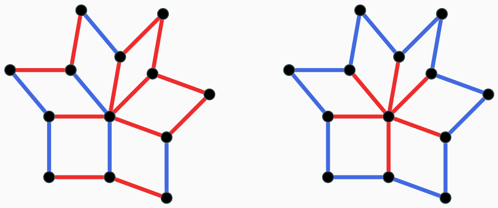


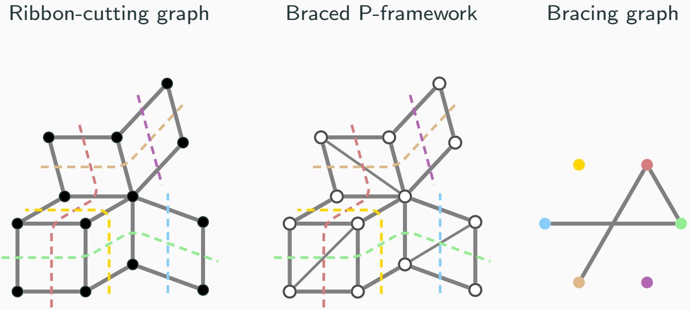
Definition

A *Penrose framework* is the P-framework obtained as the 1-skeleton of a Penrose tiling.

Definition

A NAC-coloring is called *cartesian* if no two distinct vertices are connected by a red and blue path simultaneously.




Ribbon-cutting graph
Braced P-framework
Bracing graph

Theorem (Grasegger, L., 2022)

For a braced finite P-framework (G, ρ)*, the following statements are equivalent:*

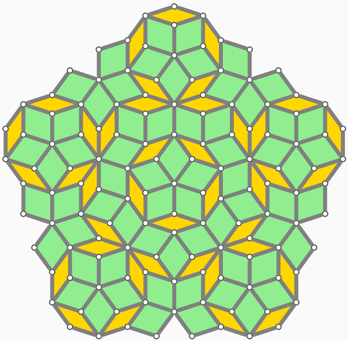
1. (G, ρ) *is flexible,*
2. G *has a cartesian NAC-coloring, and*
3. *the bracing graph of* G *is disconnected.*

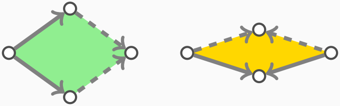
Theorem (Dewar, L., 2021+)

For a braced countably infinite P-framework (G, ρ)*, the following statements are equivalent:*

1. (G, ρ) *is flexible,*
2. G *has a cartesian NAC-coloring, and*
3. *the bracing graph of* G *is disconnected.*

Flexibility of Penrose frameworks





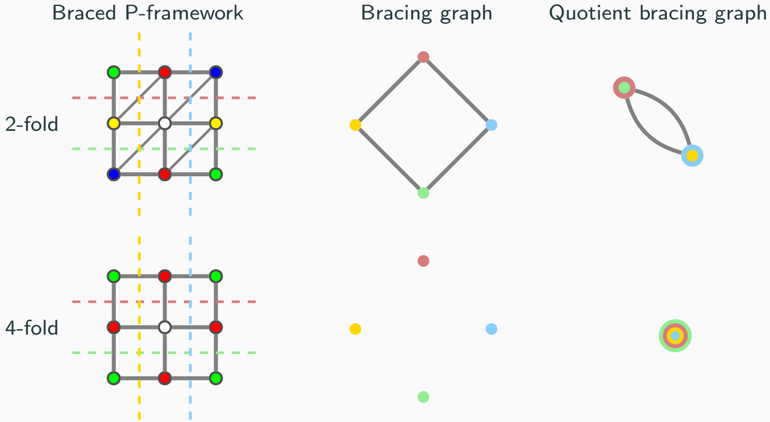
Corollary

Let (G, ρ) be a Penrose framework where we brace every rhombus in two distinct ribbons. Then (G, ρ) is rigid if and only if the ribbons "meet" each other.

Corollary

Let (G, ρ) be a Penrose framework where we brace every fat rhombus with non-zero probability s. Then (G, ρ) is rigid.

Rotationally symmetric P-frameworks



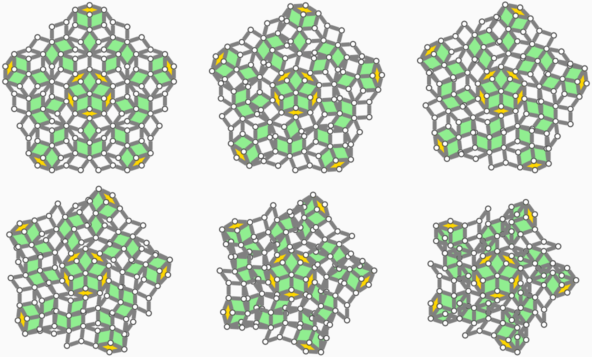
Theorem (Dewar, L., 2021+)

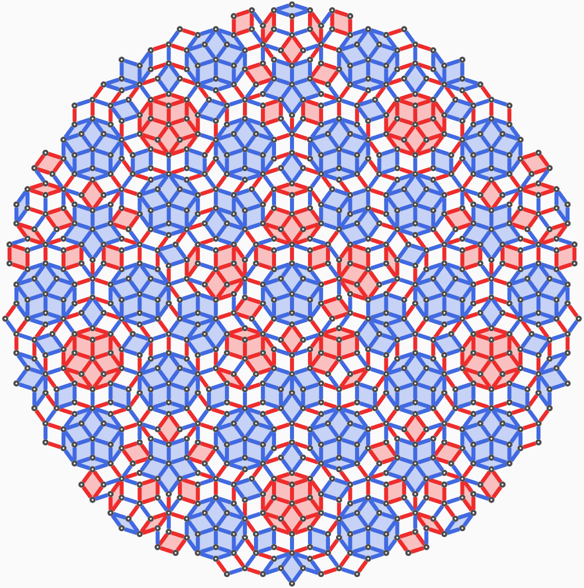
Let (G, ρ) be a braced n-fold rotationally symmetric P-framework (possibly countably infinite). Then the following are equivalent:

1. *There is a non-trivial flex of (G, ρ) that preserves the n-fold rotational symmetry.*
2. *There is a cartesian NAC-coloring of G satisfying the symmetry.*
3. *The quotient bracing graph of G is disconnected.*

Corollary

The 1-skeleton of a Penrose framework with 5-fold rotational symmetry has a non-trivial flex that preserves 5-fold rotational symmetry.





Thank you

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jan.legersky.cz/PenroseFrameworks