Stable cuts and NAC-colorings

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workshop Matroids, Rigidity, and Algebraic Statistics

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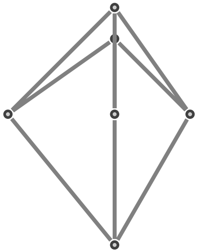

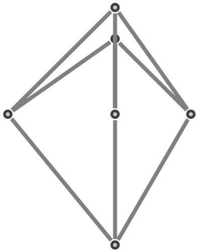 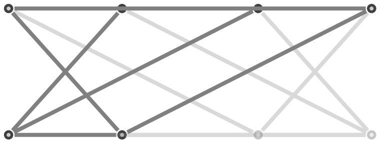

Dixon (1900): two constructions of flexible $K_{3,3}$

Walter and Husty (2007): there is no other one.

Rigidity theory

Definition

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A *flex* of the *framework* (G, p) is a continuous path $t \mapsto p_t$, $t \in [0, 1)$, in the space of realizations of G such that $p_0 = p$ and for all $t \in [0, 1)$ and all edges $uv \in E$

$$\|p_t(u) - p_t(v)\| = \|p(u) - p(v)\| .$$

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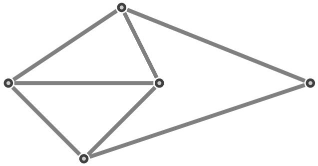
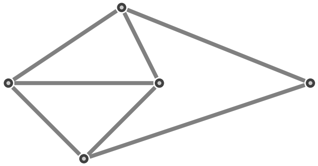
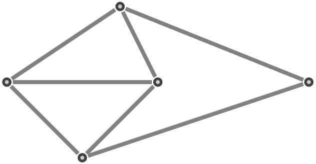
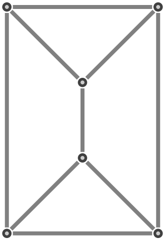
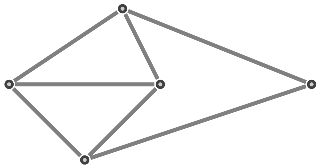
The framework (G, p) is *flexible* if it has a non-trivial flex. Otherwise it is called *rigid*.

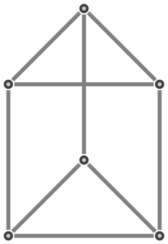


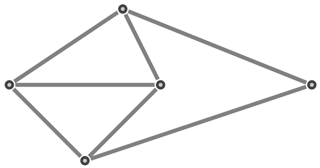


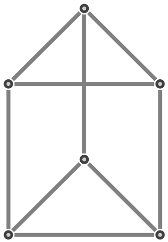












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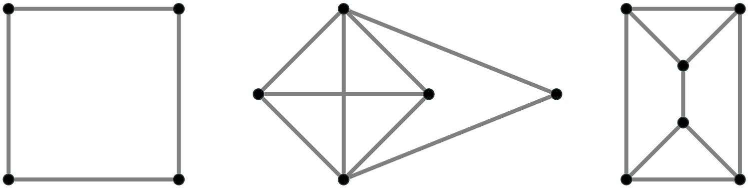


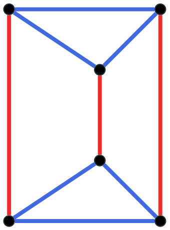
Theorem (Pollaczek-Geiringer 1927, Laman 1970)
A graph $G = (V, E)$ is minimally rigid if and only if $|E| = 2|V| - 3$ and $|E'| \leq 2|V'| - 3$ for every subgraph (V', E') of G with at least two vertices.

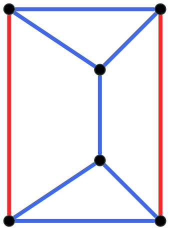
Pebble game algorithms allow to check the condition above in polynomial time.

The existence of flexible frameworks

Definition

A coloring of edges $\delta : E \to \{\text{blue, red}\}$ is called a *NAC-coloring*, if it is surjective and for every cycle in G, either all edges in the cycle have the same color, or there are at least two blue and two red edges in the cycle.





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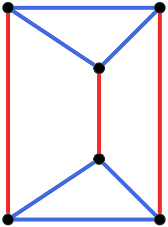

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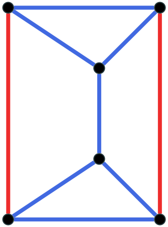

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Theorem (Grasegger, L., Schicho, 2019, Dewar, L., 2023)
A connected graph has a quasi-injective flexible realization in the plane if and only if it has a NAC-coloring.

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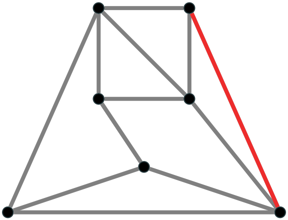

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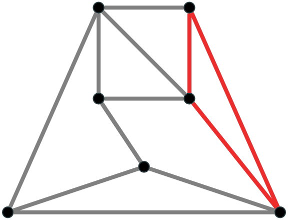

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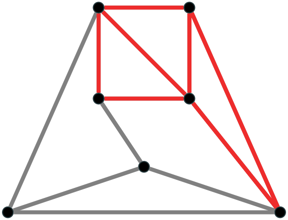

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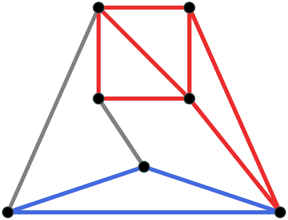

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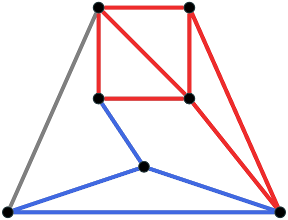

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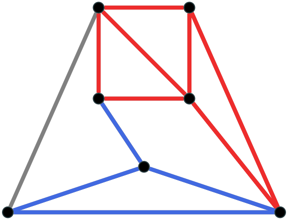

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Theorem (Garamvölgyi, 2022)

The existence of a NAC-coloring of a graph is NP-complete.

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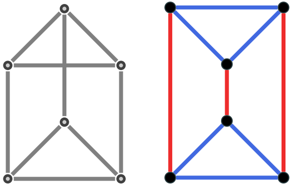
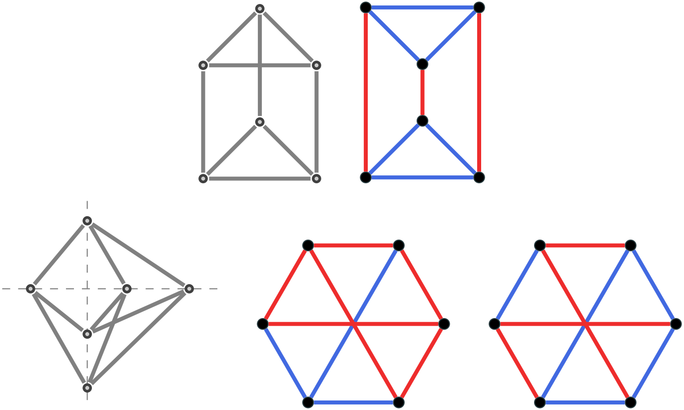
Theorem (Laštovička, L., 2024+)

The existence of a NAC-coloring is NP-complete on graphs with the maximum degree five, resp. $\leq (2+\varepsilon)|V|$ *edges.*





Minimally rigid graphs with flexible realizations

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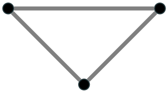

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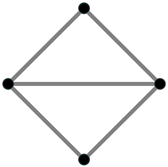

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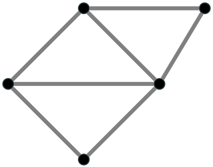

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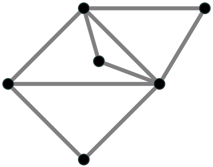

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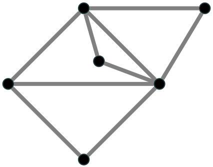


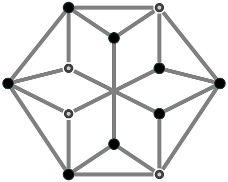
Theorem

Let G be a minimally rigid graph. The following are equivalent:

1. *G has a NAC-colouring,*
2. *G is not a 2-tree, and*
3. *G has a flexible quasi-injective realisation in the plane.*

Definition

A *stable cut* S of a connected graph G is a set of vertices such that S is stable (independent) and $G - S$ is disconnected.



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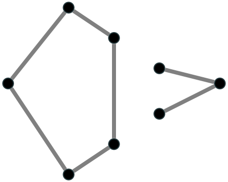

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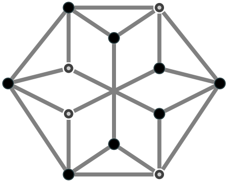

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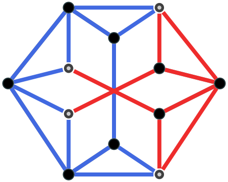

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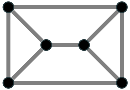
Theorem (Le, Pfender, 2013, Rauch, Rautenbach, 2024+)
Let $G = (V, E)$ be a graph such that $|E| = 2|V| - 3$. Then either G has a stable cut or G belongs to class $\mathcal{G}$:

1. *K_2, K_3 and the 3-prism are in $\mathcal{G}$, and*
2. *if $G_1, G_2 \in \mathcal{G}$, then the graph obtained by gluing G_1 and G_2 along an edge or a 3-cycle is in $\mathcal{G}$.*



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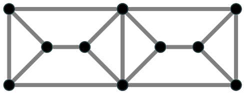

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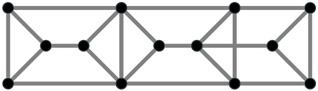

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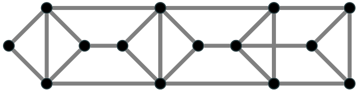

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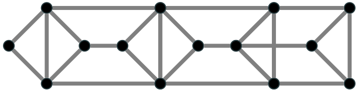

Theorem (Chen, Yu, 2002)

Every graph with at most $2|V| - 4$ edges has a stable cut.

Stable cuts in flexible graphs

Theorem (Whiteley, 1983)

A graph is rigid in the plane if and only if it is rigid on the sphere.

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A connected graph has a spherical flexible realisation on the sphere if and only if it has a stable cut.

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Corollary

Every flexible graph has a stable cut.

Theorem (Chen, Yu, 2002)

Every 2-connected graph with at most $2|V| - 4$ *edges has a stable cut which avoids a given vertex.*

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Let G be a 2-connected flexible graph. Suppose u, v are two vertices of G such that no rigid component of G contains both of them. Then there is a stable cut S of G that separates u and v.

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It can be found in time $O(|V|^3)$.

Number of NAC-colorings

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Let G be a 2-connected flexible graph with m rigid components. Then

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Lemma

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Theorem

Let G be any graph on n vertices. Then

$$\mathrm{NAC}_{\#}(G) \leq \binom{2n-5}{n-2} \sim \frac{2^{2n-5}}{\sqrt{\pi n}} .$$

- If T is an n-vertex tree, then $\mathrm{NAC}_{\#}(T) = 2^{n-2} - 1$.
- If C_n is the n-vertex cycle, then $\mathrm{NAC}_{\#}(C_n) = 2^{n-1} - (n+1)$.

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- If K_{n_1,n_2} is the complete bipartite graph on $n_1 + n_2$ vertices, then $\mathrm{NAC}_{\#}(K_{n_1,n_2}) = 2^{n_1+n_2-2} - 1$.

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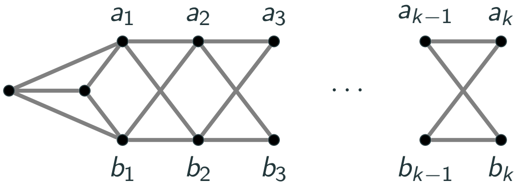
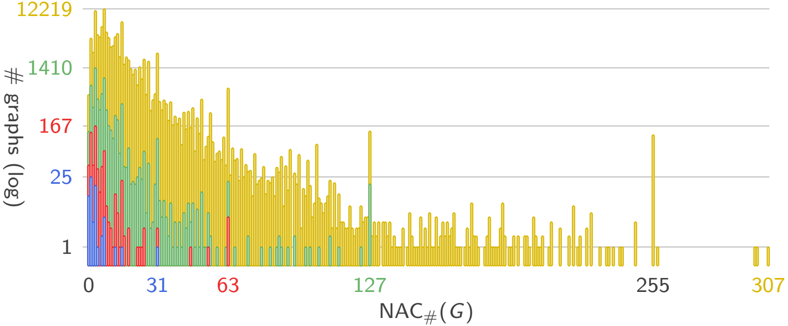
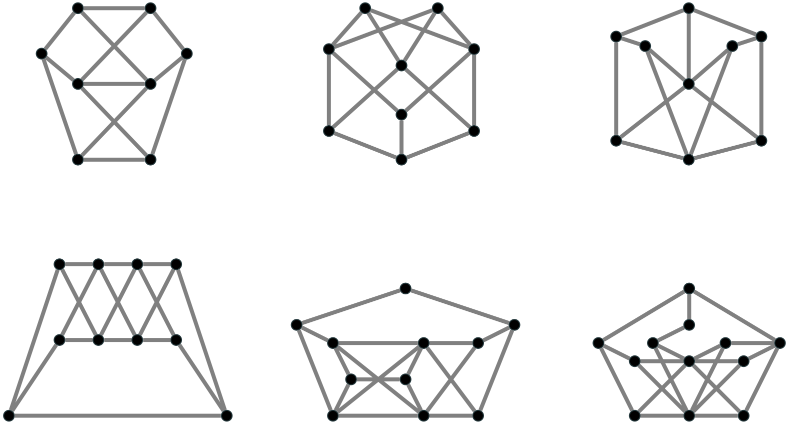
- If C_n is the n-vertex cycle, then $\mathrm{NAC}_{\#}(C_n) = 2^{n-1} - (n+1)$.
- If K_{n_1,n_2} is the complete bipartite graph on $n_1 + n_2$ vertices, then $\mathrm{NAC}_{\#}(K_{n_1,n_2}) = 2^{n_1+n_2-2} - 1$.
- If G_k is the minimally rigid graph below with $n = 2k + 2$ vertices, then $\mathrm{NAC}_{\#}(G_k) = 2^{n-4} - 1$.





The maximum $NAC_{\#}(\cdot)$ among minimally rigid graphs on 11 and 12 vertices is 638 and 1461 respectively.



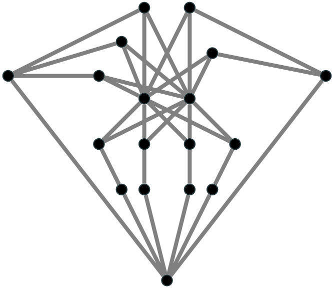
Lemma

Let G be a graph obtained by gluing k copies of a graph H along an edge. Then

$$\mathrm{NAC}_{\#}(G) = (\mathrm{NAC}_{\#}(H) + 1)^k - 1 = (\mathrm{NAC}_{\#}(H) + 1)^{\frac{|V(G)|-2}{|V(H)|-2}} - 1\,.$$

Corollary

There is an infinite family of minimally rigid graphs H_k such that $\mathrm{NAC}_{\#}(H_k) = \omega\left(2.13^{|V(H_k)|}\right)$.



18-vertex minimally-rigid graph H with $\mathrm{NAC}_{\#}(H) = 180\,607$, computed using code by Petr Laštovička.

PyRigi

Python package for rigidity and flexibility of bar-joint frameworks

https://pyrigi.github.io/PyRigi/

Open problems

- Minimally rigid graphs G with $\mathrm{NAC}_{\#}(G) = 1$.

- Minimally rigid graphs G with $\mathrm{NAC}_{\#}(G) = 1$.
- NAC-colorings/stable cuts of graphs with $2|V| - 2$ edges.

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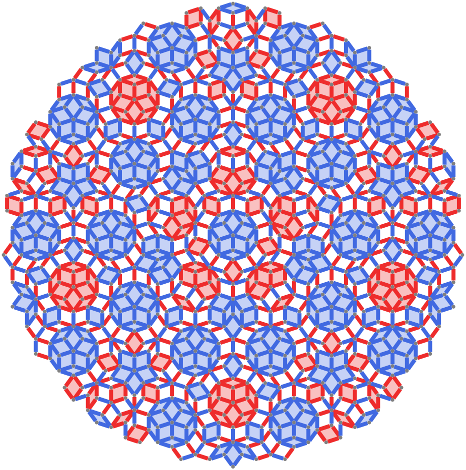
- NAC-colorings/stable cuts of graphs with $2|V| - 2$ edges.
- The asymptotic behavior of $\text{NAC}_{\#}(\cdot)$.



Thank you

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