Paradoxical Mobility of $K_{3,3}$ Revisited

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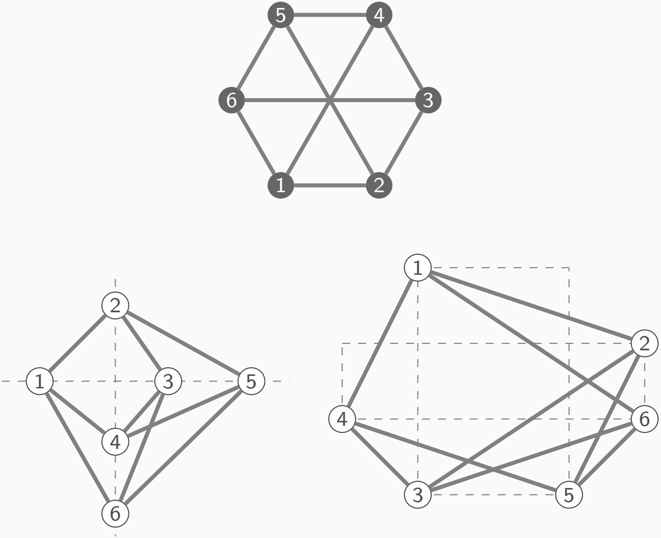


Dixon (1899), Walter and Husty (2007)

Proper flexible labeling

An edge labeling $\lambda : E \to \mathbb{R}_+$ of $K_{3,3} = (V, E)$ is called *proper flexible* if the system

$$\begin{aligned}(x_{\bar{u}}, y_{\bar{u}}) &= (0, 0) \\ (x_{\bar{v}}, y_{\bar{v}}) &= (\lambda_{\bar{u}\bar{v}}, 0) \\ (x_u - x_v)^2 + (y_u - y_v)^2 &= \lambda_{uv}^2, \quad \forall\, uv \in E\,.\end{aligned}$$

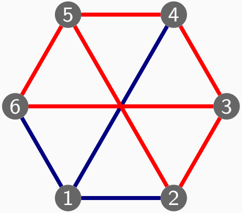
has a 1-dimensional irreducible component, called an *algebraic motion*, such that

$$(x_u - x_v)^2 + (y_u - y_v)^2 \neq 0, \quad \forall\, uv \notin E\,.$$

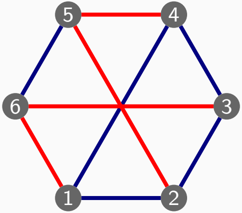
NAC-colorings

Definition

A coloring of edges $\delta : E \to \{\text{blue, red}\}$ is called a *NAC-coloring*, if it is surjective and for every cycle in G, either all edges in the cycle have the same color, or there are at least two blue and two red edges in the cycle.



ω_1



ϵ_{56}

Active NAC-colorings

$$\begin{aligned}\lambda_{uv}^2 &= (x_v - x_u)^2 + (y_v - y_u)^2 \\ &= \underbrace{((x_v - x_u) + i(y_v - y_u))}_{W_{u,v}}\underbrace{((x_v - x_u) - i(y_v - y_u))}_{Z_{u,v}}\end{aligned}$$

Lemma (GLS)

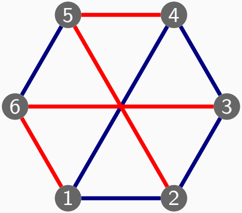
Let λ be a flexible labeling of a graph G. Let $\mathcal{C}$ be an algebraic motion of (G, λ). If $\alpha \in \mathbb{Q}$ and ν is a valuation of the complex function field of $\mathcal{C}$ such that there exists edges $\bar{u}\bar{v}, \hat{u}\hat{v}$ with $\nu(W_{\bar{u},\bar{v}}) = \alpha$ and $\nu(W_{\hat{u},\hat{v}}) > \alpha$, then $\delta : E_G \to \{red, blue\}$ given by

$$\begin{aligned}\delta(uv) = red &\iff \nu(W_{u,v}) > \alpha\,, \\ \delta(uv) = blue &\iff \nu(W_{u,v}) \leq \alpha\,.\end{aligned}$$

is a NAC-coloring, called active.

Quadrilateral	Motion	Active NAC-colorings
Rhombus	parallel	
	degenerate	 resp. 
Parallelogram		
Antiparallelogram		
Deltoid	nondegenerate	
	degenerate	
General		

Orthogonal diagonals



ϵ_{56}

Lemma (GLS)

If ϵ_{56} is an active NAC-coloring of an algebraic motion of $(K_{3,3}, \lambda)$, then

$$\lambda_{12}^2 + \lambda_{34}^2 = \lambda_{23}^2 + \lambda_{14}^2 \,,$$

namely, the 4-cycle $(1, 2, 3, 4)$ has orthogonal diagonals.

Ramification formula

Theorem (GLS)

Let $\mathcal{C}$ be an algebraic motion of (G, λ) with the set of active NAC-colorings N. There exist $\mu_\delta \in \mathbb{Z}_{\geq 0}$ for all NAC-colorings δ of G such that:

1. $\mu_\delta \neq 0$ *if and only if* $\delta \in N$, *and*
2. *for every 4-cycle* (V_i, E_i) *of* G, *there exists a positive integer* d_i *such that*

$$\sum_{\substack{\delta \in \mathrm{NAC}_G \\ \delta|_{E_i} = \delta'}} \mu_\delta = d_i \qquad \text{for all } \delta' \in \{\delta|_{E_i} : \delta \in N\}\,.$$

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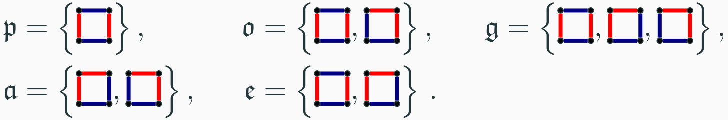

Classification of motions of $K_{3,3}$

- Find all possible types of motions of 4-cycles with consistent μ_δ's – 112 out of
  - $32\,768 = 2^{15}$ subsets of NAC-colorings, or
  - $1\,953\,125 = 5^9$ motion types of 4-cycles, or
  - $3\,075$ motion types of 4-cycles iteratively

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- Remove combinations with coinciding vertices (due to edge lengths, perpendicular diagonals)
  - 34 left

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- Identify symmetric cases
  - 4 classes

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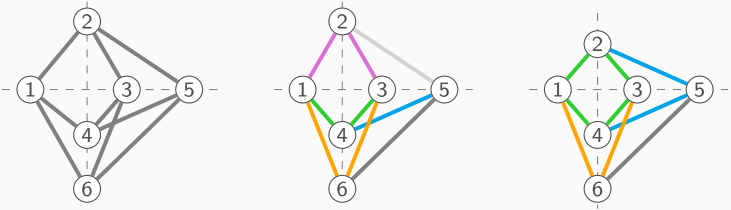
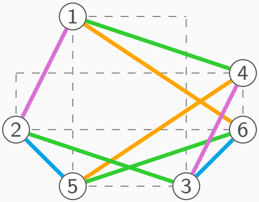
Implementation – SageMath package `FlexRiLoG` (`https://github.com/Legersky/flexrilog`)



4-cycles	active NAC-colorings	#	
$\mathfrak{ggggggggg}$	$\mathrm{NAC}_{K_{3,3}}$	1	
$\mathfrak{ooogggggg}$	$\{\epsilon_{12}, \epsilon_{23}, \epsilon_{34}, \epsilon_{14}, \epsilon_{16}, \epsilon_{36}, \omega_1, \omega_3\}$	6	Dixon I
$\mathfrak{pooggogge}$	$\{\epsilon_{12}, \epsilon_{23}, \epsilon_{34}, \epsilon_{14}\}$	9	
$\mathfrak{pgggaggag}$	$\{\epsilon_{12}, \epsilon_{34}, \omega_5, \omega_6\}$	18	Dixon II



Computer-free proof

... for every 4-cycle (V_i, E_i) *of* G*, there exists a positive integer* d_i *such that*

$$\sum_{\substack{\delta \in \mathrm{NAC}_G \\ \delta|_{E_i} = \delta'}} \mu_\delta = d_i \qquad \textit{for all } \delta' \in \{\delta|_{E_i} : \delta \in N\}\,.$$

The number d_i is actually the degree of the projection of the algebraic motion to the i-th 4-cycle.

$\Longrightarrow$ can be determined (assuming that the motion is not Dixon I) and used to decrease the number of cases to be checked (26).

Thank you

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