Flexible and Rigid Labelings of Graphs

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Outline

- Existence of Flexible Labelings
- Movable Graphs
- On the Classification of Motions
- Number of Real Realizations compatible with a Rigid Labeling
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- **Movable Graphs**

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- **Number of Real Realizations compatible with a Rigid Labeling**
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- On the Classification of Motions

- Number of Real Realizations compatible with a Rigid Labeling
Let $\lambda : E_G \to \mathbb{R}_+$ be an edge labeling of a graph $G = (V_G, E_G)$. A realization $\rho : V_G \to \mathbb{R}^2$ is compatible with $\lambda$ if $\|\rho(u) - \rho(v)\| = \lambda(uv)$ for all edges $uv$ in $E_G$. The labeling $\lambda$ is called

- proper flexible if there are infinitely many non-congruent injective compatible realizations,
- rigid if the number of non-congruent compatible realizations is positive and finite.

A graph is called movable if it has a proper flexible labeling.
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$$\|\rho(u) - \rho(v)\| = \lambda(uv)$$

for all edges $uv$ in $E_G$.

The labeling $\lambda$ is called

- **flexible** if there are infinitely many non-congruent compatible realizations, or
- **rigid** if the number of non-congruent compatible realizations is positive and finite.
Flexible and rigid labelings

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- **proper flexible** if there are infinitely many non-congruent injective compatible realizations, or
- **rigid** if the number of non-congruent compatible realizations is positive and finite.

A graph is called *movable* if it has a proper flexible labeling.
Algebraic formulation

\[(x_u, y_u) = (0, 0)\]

\[(x_v, y_v) = (\lambda (u\bar{v}), 0)\]

\[(x_u - x_v)^2 + (y_u - y_v)^2 = \lambda (uv)^2, \quad \forall uv \in E_G\]
Algebraic formulation

\[(x_\bar{u}, y_\bar{u}) = (0, 0)\]

\[(x_\bar{v}, y_\bar{v}) = (\lambda(\bar{u}\bar{v}), 0)\]

\[(x_u - x_v)^2 + (y_u - y_v)^2 = \lambda(\bar{u}\bar{v})^2, \quad \forall \ uv \in E_G\]

- only isolated solutions \(\implies\) \(\lambda\) is rigid,
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- only isolated solutions \(\implies\) \(\lambda\) is rigid,
- infinitely many solutions \(\implies\) \(\lambda\) is flexible,
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- only isolated solutions \(\implies\) \(\lambda\) is rigid,
- infinitely many solutions \(\implies\) \(\lambda\) is flexible,
- infinitely many solutions such that
  \[(x_u - x_v)^2 + (y_u - y_v)^2 \neq 0, \quad \forall\ uv \notin E\]
  \(\implies\) \(\lambda\) is proper flexible,
Algebraic formulation

\[(x_u, y_u) = (0, 0)\]
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  \[(x_u - x_v)^2 + (y_u - y_v)^2 \neq 0\]
  \(\implies\) \(\lambda\) is proper flexible,

A 1-dimensional irreducible subset of the zero set is called an **algebraic motion**.
Laman graphs

Definition
A graph \( G \) is called Laman if \( |E_G| = 2|V_G| - 3 \), and \( |E_H| \leq 2|V_H| - 3 \) for every subgraph \( H \) of \( G \).

Theorem (Pollaczeck-Geiringer, Laman)
A labeling of a graph \( G \) induced by a generic realization of \( G \) is rigid if and only if \( G \) is spanned by a Laman graph.
Laman graphs

**Definition**
A graph $G$ is called *Laman* if $|E_G| = 2|V_G| - 3$, and $|E_H| \leq 2|V_H| - 3$ for every subgraph $H$ of $G$.

**Theorem (Pollaczek-Geiringer, Laman)**
A labeling of a graph $G$ induced by a generic realization of $G$ is rigid if and only if $G$ is spanned by a Laman graph.

Are there any Laman graphs with a (proper) flexible labeling?
Existence of Flexible Labelings
Definition
A coloring of edges \( \delta : E_G \rightarrow \{\text{blue, red}\} \) is called a \textit{NAC-coloring}, if it is surjective and for every cycle in \( G \), either all edges in the cycle have the same color, or there are at least two blue and two red edges in the cycle.
Combinatorial characterization

Theorem (GLS)
A connected graph with at least one edge has a flexible labeling if and only if it has a NAC-coloring.
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A connected graph with at least one edge has a flexible labeling if and only if it has a NAC-coloring.

⇒ no flexible labeling
Grid construction
Example

1,6,9
Grid construction II
Example II
Functions $W_{u,v}$ and $Z_{u,v}$

\[
\lambda_{uv}^2 = (x_v - x_u)^2 + (y_v - y_u)^2
= \left( (x_v - x_u) + i(y_v - y_u) \right) \left( (x_v - x_u) - i(y_v - y_u) \right)
= W_{u,v} \cdot Z_{u,v}
\]
Functions $W_{u,v}$ and $Z_{u,v}$

$$\lambda_{uv}^2 = (x_v - x_u)^2 + (y_v - y_u)^2$$

$$= ((x_v - x_u) + i(y_v - y_u))((x_v - x_u) - i(y_v - y_u))$$

For every cycle $(u_1, \ldots, u_n, u_{n+1} = u_1)$:

$$\sum_{i=1}^{n} W_{u_i, u_{i+1}} = 0 \text{ and } \sum_{i=1}^{n} Z_{u_i, u_{i+1}} = 0$$
Functions $W_{u,v}$ and $Z_{u,v}$

\[
\lambda_{uv}^2 = (x_v - x_u)^2 + (y_v - y_u)^2 \\
= \underbrace{((x_v - x_u) + i(y_v - y_u))}_{W_{u,v}} \underbrace{((x_v - x_u) - i(y_v - y_u))}_{Z_{u,v}}
\]

For every cycle $(u_1, \ldots, u_n, u_{n+1} = u_1)$:

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\sum_{i=1}^{n} W_{u_i, u_{i+1}} = 0 \quad \text{and} \quad \sum_{i=1}^{n} Z_{u_i, u_{i+1}} = 0
\]

For a valuation $\nu : F(C) \to \mathbb{Z}$ trivial on $C$:

- $\nu(W_{u,v}Z_{u,v}) = \nu(\lambda_{uv}^2)$
Functions $W_{u,v}$ and $Z_{u,v}$

\[ \lambda_{uv}^2 = (x_v - x_u)^2 + (y_v - y_u)^2 \]
\[ = ((x_v - x_u) + i(y_v - y_u))((x_v - x_u) - i(y_v - y_u)) \]

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\[ \lambda_{uv}^2 = (x_v - x_u)^2 + (y_v - y_u)^2 = (W_{u,v} + i(Z_{u,v})) (W_{u,v} - i(Z_{u,v})) \]

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\lambda_{uv}^2 = (x_v - x_u)^2 + (y_v - y_u)^2
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= \underbrace{((x_v - x_u) + i(y_v - y_u))}^{W_{u,v}} \underbrace{((x_v - x_u) - i(y_v - y_u))}^{Z_{u,v}}
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- $\nu(W_{u,v}) + \nu(Z_{u,v}) = \nu(W_{u,v}Z_{u,v}) = \nu(\lambda_{uv}^2) = 0$, and
- $W_{u_1,u_n} = \sum_{i=1}^{n-1} W_{u_i,u_{i+1}}$
Functions $W_{u,v}$ and $Z_{u,v}$

\[
\lambda_{uv}^2 = (x_v - x_u)^2 + (y_v - y_u)^2 = \left((x_v - x_u) + i(y_v - y_u)\right)\left((x_v - x_u) - i(y_v - y_u)\right)
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\]

For a valuation $\nu : F(\mathbb{C}) \to \mathbb{Z}$ trivial on $\mathbb{C}$:

1. $\nu(W_{u,v}) + \nu(Z_{u,v}) = \nu(W_{u,v}Z_{u,v}) = \nu(\lambda_{uv}^2) = 0$, and
2. $\nu(W_{u_1,u_n}) = \nu(\sum_{i=1}^{n-1} W_{u_i,u_{i+1}}) \geq \min_{i \in \{1, \ldots, n-1\}} \nu(W_{u_i,u_{i+1}})$. 
Active NAC-colorings

Lemma (GLS)

Let $\mathcal{C}$ be an algebraic motion of $(G, \lambda)$. If $\alpha \in \mathbb{Q}$ and $\nu$ is a valuation of $F(\mathcal{C})$ trivial on $\mathbb{C}$ such that there exist edges $\overline{u\nu}, \hat{u}\hat{v}$ with $\nu(W_{\overline{u},\overline{v}}) = \alpha$ and $\nu(W_{\hat{u},\hat{v}}) > \alpha$, then $\delta : E_G \to \{\text{red, blue}\}$ given by

\[
\delta(uv) = \text{red} \iff \nu(W_{u,v}) > \alpha,
\]
\[
\delta(uv) = \text{blue} \iff \nu(W_{u,v}) \leq \alpha.
\]

is a NAC-coloring, called active.
Movable Graphs
Lemma (GLS)

Let $G$ be a graph and $u, v \in V_G$ be such that $uv \notin E_G$. If there exists a $uv$-path $P$ in $G$ such that $P$ is unicolor for all NAC-colorings of $G$, then $G$ is movable if and only if $G' = (V_G, E_G \cup \{uv\})$ is movable.
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Theorem (GLS)

The maximal movable graphs with at most 8 vertices that are spanned by a Laman graph and have no vertex of degree two are the following: $K_{3,3}, K_{3,4}, K_{3,5}, K_{4,4}$ or
Embedding in $\mathbb{R}^3$

Lemma (GLS)

*If there exists an injective realization of $G$ in $\mathbb{R}^3$ such that every edge is parallel to one of the four vectors $(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, -1, -1)$, then $G$ is movable.*
Lemma (GLS)

If there exists an injective realization of $G$ in $\mathbb{R}^3$ such that every edge is parallel to one of the four vectors $(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, -1, -1)$, then $G$ is movable.

Moreover, there exists an algebraic motion of $G$ with exactly two active NAC-colorings. Two edges are parallel in the embedding $\omega$ if and only if they receive the same pair of colors in the two active NAC-colorings.
On the Classification of Motions
Classification of motions

Dixon (1899), Walter and Husty (2007)
Classification of motions

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### Active NAC-colorings of quadrilaterals

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<thead>
<tr>
<th>Quadrilateral</th>
<th>Motion</th>
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Assume a valuation that gives only one active NAC-coloring $\implies$ Laurent series parametrization.

For every cycle $C = (u_1, \ldots, u_n, u_{n+1} = u_1)$:

$$\sum_{i \in \{1, \ldots, n\}} (w_{u_i u_{i+1}} t + \text{h.o.t.}) \quad + \quad \sum_{i \in \{1, \ldots, n\}} (w_{u_i u_{i+1}} + \text{h.o.t.}) = 0.$$
Assume a valuation that gives only one active NAC-coloring $\implies$ Laurent series parametrization.

For every cycle $C = (u_1, \ldots, u_n, u_{n+1} = u_1)$:

$$\sum_{i \in \{1, \ldots, n\} \atop \delta(u_i u_{i+1}) = \text{blue}} w_{u_i u_{i+1}} = 0.$$
Assume a valuation that gives only one active NAC-coloring $\implies$ Laurent series parametrization.

For every cycle $C = (u_1, \ldots, u_n, u_{n+1} = u_1)$:

$$\sum_{i \in \{1, \ldots, n\}} \delta(u_i u_{i+1}) = \text{red}(w_{u_i u_{i+1}}) + \sum_{i \in \{1, \ldots, n\}} \delta(u_i u_{i+1}) = \text{blue} w_{u_i u_{i+1}} = 0.$$ 

For all $uv \in E_G$:

$$w_{uv} z_{uv} = \lambda_{uv}^2.$$ 

$\implies$ elimination using Gröbner basis provides an equation in $\lambda_{uv}$'s.
If a valuation yields two active NAC-colorings $\delta, \delta'$, then the set $\{(\delta(e), \delta'(e)) : e \in E_G\}$ has 3 elements.
Triangle in $Q_1$

\[
\implies \lambda_5^2 r^2 + \lambda_6^2 s^2 + (\lambda_5^2 - \lambda_5^2 - \lambda_6^2) rs = 0,
\]

\[
r = \lambda_2^2 - \lambda_2^2, \quad s = \lambda_2^2 - \lambda_2^2
\]
Triangle in $Q_1$

\[ \lambda^2_{57} r^2 + \lambda^2_{67} s^2 + (\lambda^2_{56} - \lambda^2_{57} - \lambda^2_{67}) rs = 0, \]

\[ r = \lambda^2_{24} - \lambda^2_{23}, \quad s = \lambda^2_{14} - \lambda^2_{13} \]

Considering the equation as a polynomial in $r$, the discriminant is

\[ (\lambda_{56} + \lambda_{57} + \lambda_{67})(\lambda_{56} + \lambda_{57} - \lambda_{67})(\lambda_{56} - \lambda_{57} + \lambda_{67})(\lambda_{56} - \lambda_{57} - \lambda_{67}) s^2. \]
Triangle in $Q_1$

$$\Rightarrow \lambda_{57}^2 r^2 + \lambda_{67}^2 s^2 + (\lambda_{56}^2 - \lambda_{57}^2 - \lambda_{67}^2) rs = 0,$$

$$r = \lambda_{24}^2 - \lambda_{23}^2, \ s = \lambda_{14}^2 - \lambda_{13}^2$$

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$$(\lambda_{56} + \lambda_{57} + \lambda_{67})(\lambda_{56} + \lambda_{57} - \lambda_{67})(\lambda_{56} - \lambda_{57} + \lambda_{67})(\lambda_{56} - \lambda_{57} - \lambda_{67})s^2.$$

**Theorem (GLS)**

*The vertices 5, 6 and 7 are collinear for every proper flexible labeling of $Q_1*. 

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Lemma (GLS)

If there is an active NAC-coloring $\delta$ of an algebraic motion of $(G, \lambda)$ such that a 4-cycle $(1, 2, 3, 4)$ is blue and there are red paths from 1 to 3 and from 2 to 4, then

$$\lambda_{12}^2 + \lambda_{34}^2 = \lambda_{23}^2 + \lambda_{14}^2,$$

namely, the 4-cycle $(1, 2, 3, 4)$ has orthogonal diagonals.
Theorem (GLS)

Let $\mathcal{C}$ be an algebraic motion of $(G, \lambda)$ with the set of active NAC-colorings $N$. There exist $\mu_\delta \in \mathbb{Z}_{\geq 0}$ for all NAC-colorings $\delta$ of $G$ such that:

1. $\mu_\delta \neq 0$ if and only if $\delta \in N$, and

2. for every 4-cycle $(V_i, E_i)$ of $G$, there exists a positive integer $d_i$ such that

$$\sum_{\substack{\delta \in \text{NAC}_G \\delta|_{E_i} = \delta'}} \mu_\delta = d_i \quad \text{for all } \delta' \in \{\delta|_{E_i} : \delta \in N\}.$$
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$$\begin{align*}
p &= \{\begin{array}{c}
\end{array}\}, & o &= \{\begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}\}, & g &= \{\begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}\}, \\
a &= \{\begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}\}, & e &= \{\begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}\}.
\end{align*}$$
Example

\[ \begin{align*}
\epsilon_{13} & = \mu_{\eta} = 0 \\
\epsilon_{14} & = \mu_{\psi_1} \\
\epsilon_{23} & = \mu_{\psi_2} \\
\epsilon_{24} & = \mu_{\phi_3} \\
\gamma_1 & = \mu_{\phi_4} \\
\gamma_2 & = \zeta
\end{align*} \]
Example

Antiparallelogram $\left(\begin{smallmatrix} \epsilon_1 \epsilon_2 \\ \eta \psi_1 \end{smallmatrix}\right) \Rightarrow$
Example

Antiparallelogram \((\square, \square)\) \(\implies\)

\[\mu_{\epsilon_{13}} = \mu_{\gamma_1} = \mu_\eta = 0\]
Antiparallelogram \( \square \leftrightarrow \square \) \( \implies \)

\[
\mu_{\epsilon_{13}} = \mu_{\gamma_1} = \mu_{\eta} = 0
\]

\[
\mu_{\epsilon_{14}} + \mu_{\psi_1}
\]
Example

Antiparallelogram $\left(\square, \square\right)$ \implies

$$\mu_{\epsilon_{13}} = \mu_{\gamma_1} = \mu_\eta = 0$$

$$\mu_{\epsilon_{14}} + \mu_{\psi_1} = \mu_{\epsilon_{23}} + \mu_{\gamma_2} + \mu_{\phi_3} + \mu_\zeta$$
Classification of motions

- Find all possible types of motions of quadrilaterals with consistent $\mu_\delta$’s

Implementation – SageMath package FlexRiLoG (https://github.com/Legersky/flexrilog)
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- Remove combinations with coinciding vertices (due to edge lengths, perpendicular diagonals)

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- Check if there is a proper flexible labeling satisfying the necessary conditions
Classification of motions

- Find all possible types of motions of quadrilaterals with consistent $\mu_\delta$’s
- Remove combinations with coinciding vertices (due to edge lengths, perpendicular diagonals)
- Identify symmetric cases
- Compute necessary conditions for $\lambda_{uv}$’s using leading coefficient systems
- Check if there is a proper flexible labeling satisfying the necessary conditions

Implementation – SageMath package FlexRiLoG
(https://github.com/Legersky/flexrilog)
### Classification of motions of $K_{3,3}$

<table>
<thead>
<tr>
<th>4-cycles</th>
<th>active NAC-colorings</th>
<th>#</th>
</tr>
</thead>
<tbody>
<tr>
<td>gggggggggg</td>
<td>NAC$<em>{K</em>{3,3}}$</td>
<td>1</td>
</tr>
<tr>
<td>oooogggggg</td>
<td>{\epsilon_{12}, \epsilon_{23}, \epsilon_{34}, \epsilon_{14}, \epsilon_{16}, \epsilon_{36}, \omega_1, \omega_3}</td>
<td>6</td>
</tr>
<tr>
<td>pooggogge</td>
<td>{\epsilon_{12}, \epsilon_{23}, \epsilon_{34}, \epsilon_{14}}</td>
<td>9</td>
</tr>
<tr>
<td>pggagagggg</td>
<td>{\epsilon_{12}, \epsilon_{34}, \omega_5, \omega_6}</td>
<td>18</td>
</tr>
</tbody>
</table>

![Graphs showing different motions of $K_{3,3}$](image)
## Classification of motions of $Q_1$

<table>
<thead>
<tr>
<th>4-cycles</th>
<th>active NAC-colorings</th>
<th>#</th>
<th>type</th>
<th>dim.</th>
</tr>
</thead>
<tbody>
<tr>
<td>pggpgpgg</td>
<td>{\epsilon_{13}, \epsilon_{24}, \eta}</td>
<td>2</td>
<td>I</td>
<td>4</td>
</tr>
<tr>
<td>poapope</td>
<td>{\epsilon_{13}, \eta}</td>
<td>4</td>
<td>$\subset I, IV_-, V, VI$</td>
<td>2</td>
</tr>
<tr>
<td>peepapa</td>
<td>{\epsilon_{13}, \epsilon_{24}}</td>
<td>2</td>
<td>$\subset I, II, III$</td>
<td>2</td>
</tr>
<tr>
<td>ogggggggg</td>
<td>{\epsilon_{ij}, \gamma_1, \gamma_2, \psi_1, \psi_2}</td>
<td>1</td>
<td>$II_- \cup II_+$</td>
<td>5</td>
</tr>
<tr>
<td>peegggg</td>
<td>{\epsilon_{13}, \epsilon_{14}, \epsilon_{23}, \epsilon_{24}}</td>
<td>1</td>
<td>$\subset II_-, II_+$</td>
<td>4</td>
</tr>
<tr>
<td>oggpgga</td>
<td>{\epsilon_{13}, \epsilon_{24}, \gamma_1, \psi_2}</td>
<td>4</td>
<td>$\subset II_-$</td>
<td>3</td>
</tr>
<tr>
<td>oggegge</td>
<td>{\epsilon_{13}, \epsilon_{23}, \gamma_1, \gamma_2}</td>
<td>2</td>
<td>$\subset II_-, \text{deg.}$</td>
<td>2</td>
</tr>
<tr>
<td>ogggaga</td>
<td>{\epsilon_{13}, \epsilon_{24}, \psi_1, \psi_2, \zeta}</td>
<td>2</td>
<td>III</td>
<td>3</td>
</tr>
<tr>
<td>ggapgggg</td>
<td>{\epsilon_{13}, \eta, \phi_4, \psi_2}</td>
<td>4</td>
<td>$IV_- \cup IV_+$</td>
<td>4</td>
</tr>
<tr>
<td>ggaegpe</td>
<td>{\epsilon_{13}, \eta, \gamma_2, \phi_3}</td>
<td>4</td>
<td>V</td>
<td>3</td>
</tr>
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</table>
Number of Real Realizations compatible with a Rigid Labeling
How many realizations of a Laman graph are compatible with a given rigid labeling?

\[(x_u, y_u) = (0, 0)\]
\[(x_v, y_v) = (\lambda(\bar{u} \bar{v}), 0)\]
\[(x_u - x_v)^2 + (y_u - y_v)^2 = \lambda(\bar{u} \bar{v})^2, \quad \forall \ uv \in E_G\]
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Goal — specify edge lengths with many real solutions.
Laman graphs with many real realizations

- 6 vertices: Borcea and Streinu ’04
- 7 vertices: Emiris and Moroz ’11
- Capco, Gallet, Grasegger, Koutschan, Lubbes and Schicho ’18

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<tr>
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<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
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<tbody>
<tr>
<td>minimum</td>
<td>16</td>
<td>32</td>
<td>64</td>
<td>128</td>
<td>256</td>
<td>512</td>
<td>1024</td>
</tr>
<tr>
<td>maximum ((C))</td>
<td>24</td>
<td>56</td>
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<td>344</td>
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<td>2288</td>
<td>6180</td>
</tr>
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3D – Geiringer graphs

- 6 vertices: Emiris, Tsigaridas and Varvitsiotis ’13
- Grasegger, Koutschan, Tsigaridas ’18

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<tr>
<td>minimum ((\mathbb{C}))</td>
<td>8</td>
<td>16</td>
<td>24</td>
<td>48</td>
<td>76</td>
</tr>
<tr>
<td>maximum ((\mathbb{C}))</td>
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<td>48</td>
<td>160</td>
<td>640</td>
<td>2560</td>
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<td>48</td>
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\[ G_{3,7}^{\text{max}} \]

\[ G_{3,8}^{\text{max}} \]
Removing an edge $uc$ yields a flexible structure.

The curve traced by the vertex $c$ is called a *coupler curve*. 
Removing an edge $uc$ yields a flexible structure.

The curve traced by the vertex $c$ is called a *coupler curve*.
Example
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- Movable Graphs
- On the Classification of Motions
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- Realizations on the Sphere
  - NAP-colorings, classification of motions of $K_{3,3}$
Thank you

jan.legersky@risc.jku.at
jan.legersky.cz