On the Classification of Motions of Paradoxically Movable Graphs

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joint work with Josef Schicho and Georg Grasegger

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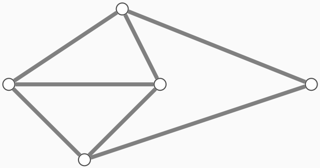
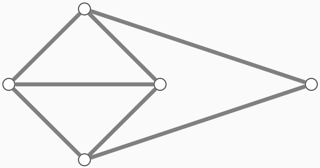
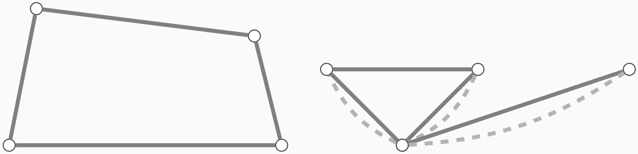
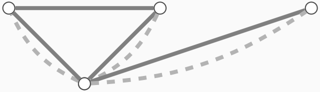
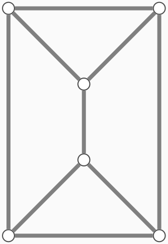
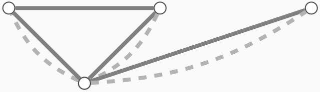


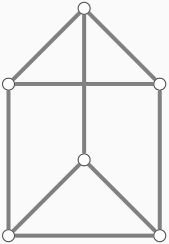


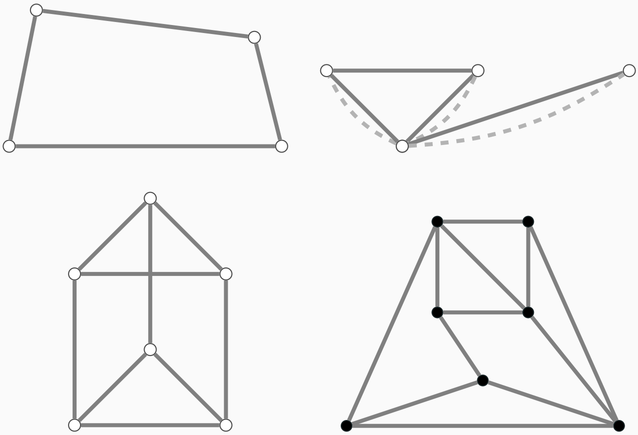


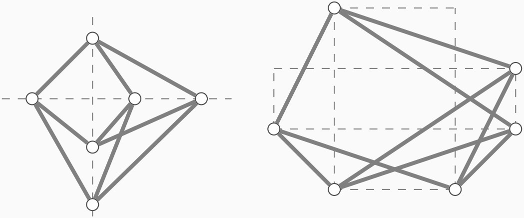




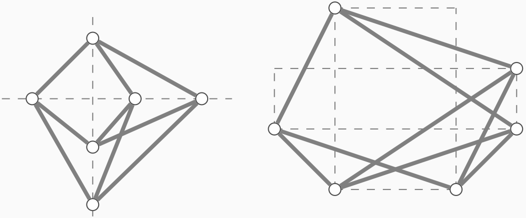








Dixon (1899), Walter and Husty (2007)



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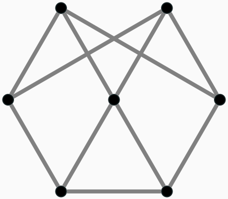

Q_1

Flexible and rigid labelings

Let $\lambda : E_G \to \mathbb{R}_+$ be an edge labeling of a graph $G = (V_G, E_G)$. A *realization* $\rho : V_G \to \mathbb{R}^2$ is compatible with λ if $\|\rho(u) - \rho(v)\| = \lambda(uv)$ for all edges uv in E_G.

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The labeling λ is called

- *proper flexible* if there are infinitely many non-congruent injective compatible realizations, or
- *rigid* if the number of non-congruent compatible realizations is positive and finite.

A graph is called *movable* if it has a proper flexible labeling.

Algebraic formulation

$$(x_{\bar{u}}, y_{\bar{u}}) = (0, 0)$$
$$(x_{\bar{v}}, y_{\bar{v}}) = (\lambda(\bar{u}\bar{v}), 0)$$
$$(x_u - x_v)^2 + (y_u - y_v)^2 = \lambda(uv)^2, \quad \forall\, uv \in E_G$$

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$\implies \lambda$ is proper flexible,

An irreducible curve in the zero set is called an *algebraic motion*.

Laman graphs

Definition

A graph G is called *Laman* if $|E_G| = 2|V_G| - 3$, and $|E_H| \leq 2|V_H| - 3$ for every subgraph H of G s.t. $|V_H| \geq 2$.

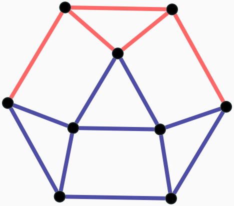
Theorem (Pollaczek-Geiringer, Laman)

A labeling of a graph G induced by a generic realization of G is rigid if and only if G is spanned by a Laman graph.

NAC-colorings

Definition

A coloring of edges $\delta : E_G \to \{\text{blue}, \text{red}\}$ is called a *NAC-coloring*, if it is surjective and for every cycle in G, either all edges in the cycle have the same color, or there are at least two blue and two red edges in the cycle.



Combinatorial characterization

Theorem (GLS)

A connected graph with at least one edge has a flexible labeling if and only if it has a NAC-coloring.

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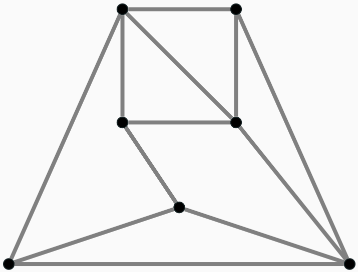

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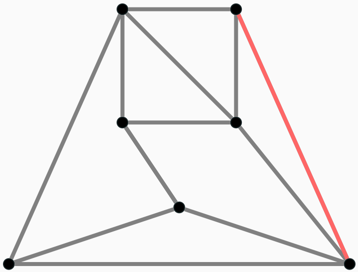

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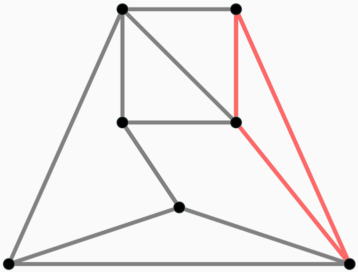

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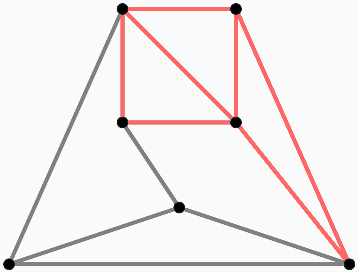

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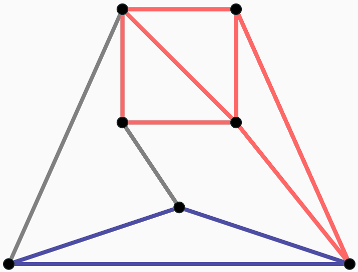

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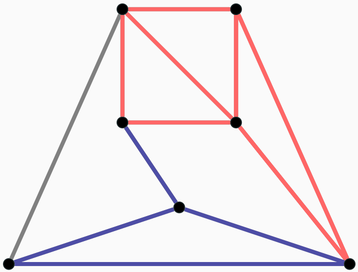

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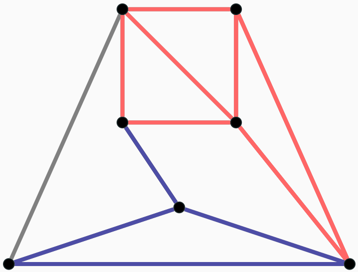
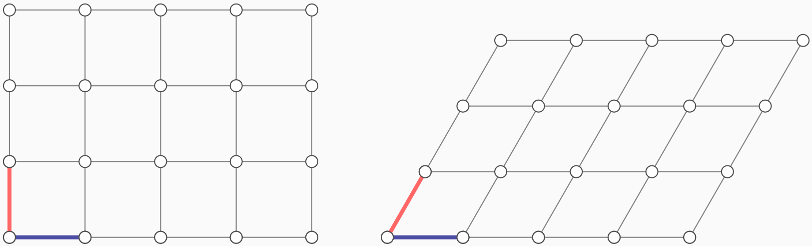
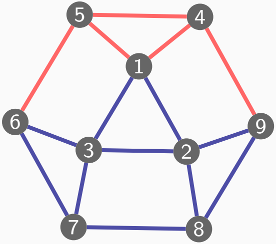


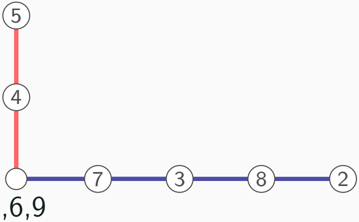
$\implies$ no flexible labeling

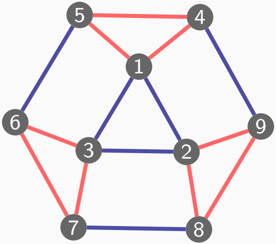
Grid construction

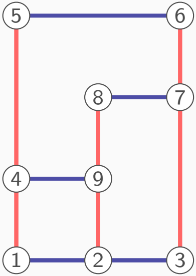


Example

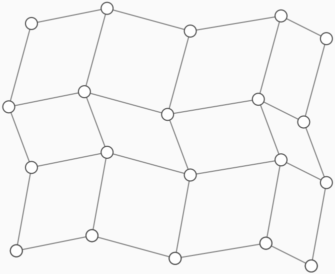


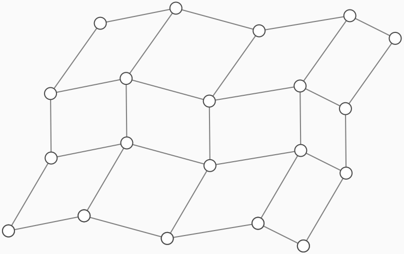






Grid construction II





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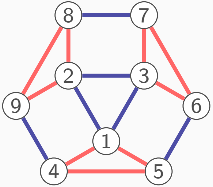

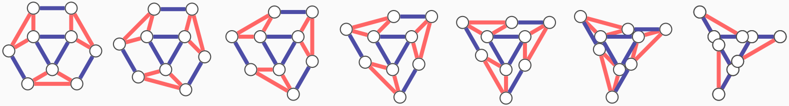

Functions $W_{u,v}$ and $Z_{u,v}$

$$\begin{aligned}\lambda_{uv}^2 &= (x_v - x_u)^2 + (y_v - y_u)^2 \\ &= \underbrace{((x_v - x_u) + i(y_v - y_u))}_{W_{u,v}}\underbrace{((x_v - x_u) - i(y_v - y_u))}_{Z_{u,v}}\end{aligned}$$

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For every cycle $(u_1, \ldots, u_n, u_{n+1} = u_1)$:

$$\sum_{i=1}^{n} W_{u_i,u_{i+1}} = 0 \quad \text{and} \quad \sum_{i=1}^{n} Z_{u_i,u_{i+1}} = 0$$

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Active NAC-colorings

Lemma (GLS)

Let $\mathcal{C}$ be an algebraic motion of (G, λ). If $\alpha \in \mathbb{Q}$ and ν is a valuation of $F(\mathcal{C})$ trivial on $\mathbb{C}$ such that there exist edges $\bar{u}\bar{v}, \hat{u}\hat{v}$ with $\nu(W_{\bar{u},\bar{v}}) = \alpha$ and $\nu(W_{\hat{u},\hat{v}}) > \alpha$, then $\delta : E_G \to \{red, blue\}$ given by

$$\delta(uv) = red \iff \nu(W_{u,v}) > \alpha\,,$$
$$\delta(uv) = blue \iff \nu(W_{u,v}) \leq \alpha\,.$$

is a NAC-coloring, called active.

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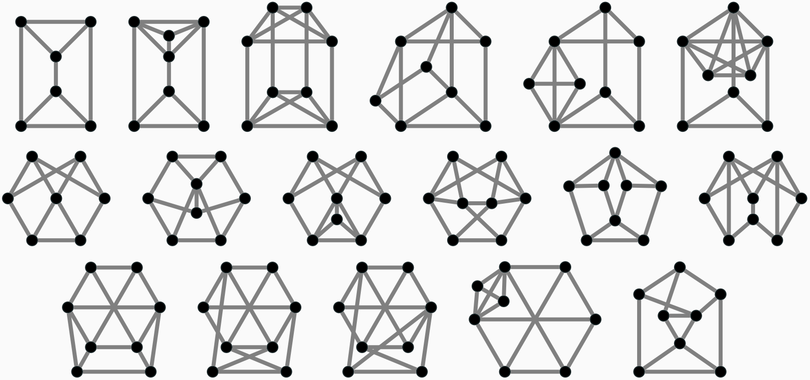
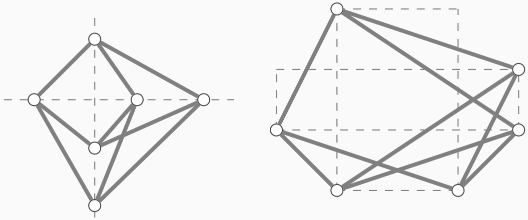
Movable graphs up to 8 vertices

Theorem (GLS)

The maximal movable graphs with at most 8 vertices that are spanned by a Laman graph and have no vertex of degree two are the following: $K_{3,3}, K_{3,4}, K_{3,5}, K_{4,4}$ *or*



Classification of motions



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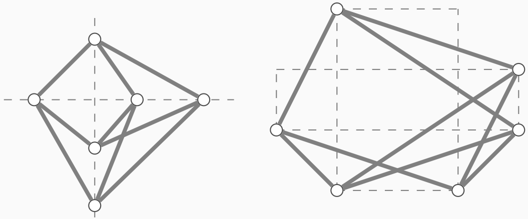

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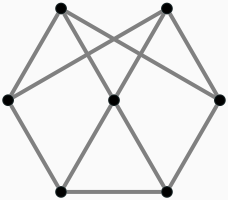

Q_1

Active NAC-colorings of quadrilaterals

Quadrilateral	**Motion**	**Active NAC-colorings**
Rhombus	parallel	
	degenerate	 resp. 
Parallelogram		
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Deltoid	nondegenerate	 
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General		  

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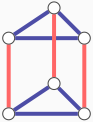

Leading coefficient system

Assume a valuation that gives only one active NAC-coloring
$\implies$ Laurent series parametrization.

For every cycle $C = (u_1, \ldots, u_n, u_{n+1} = u_1)$:

$$\sum_{\substack{i \in \{1,\ldots,n\} \\ \delta(u_i u_{i+1}) = \text{red}}} \underbrace{\left(w_{u_i u_{i+1}} t + \text{h.o.t.}\right)}_{W_{u_i, u_{i+1}}} + \sum_{\substack{i \in \{1,\ldots,n\} \\ \delta(u_i u_{i+1}) = \text{blue}}} \underbrace{\left(w_{u_i u_{i+1}} + \text{h.o.t.}\right)}_{W_{u_i, u_{i+1}}} = 0 \,.$$

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$\implies$ Laurent series parametrization.

For every cycle $C = (u_1, \ldots, u_n, u_{n+1} = u_1)$:

$$\sum_{\substack{i \in \{1,\ldots,n\} \\ \delta(u_i u_{i+1}) = \text{red}}} z_{u_i u_{i+1}} = \sum_{\substack{i \in \{1,\ldots,n\} \\ \delta(u_i u_{i+1}) = \text{blue}}} w_{u_i u_{i+1}} = 0\,.$$

Leading coefficient system

Assume a valuation that gives only one active NAC-coloring
$\implies$ Laurent series parametrization.

For every cycle $C = (u_1, \ldots, u_n, u_{n+1} = u_1)$:

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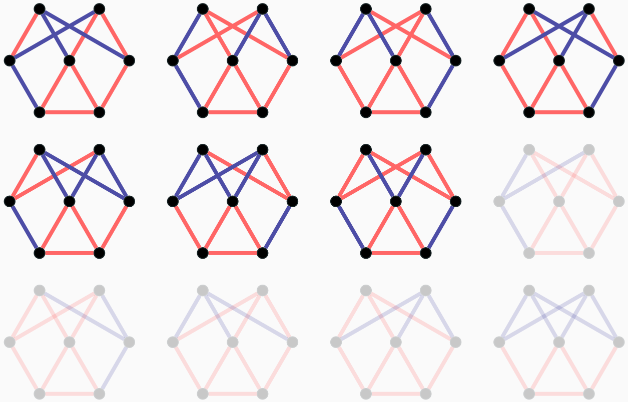
For all $uv \in E_G$:

$$w_{uv} z_{uv} = \lambda_{uv}^2\,.$$

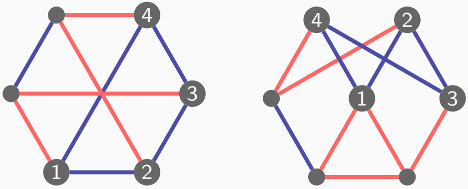
$\implies$ elimination using Gröbner basis gives an equation in λ_{uv}'s.

Singleton NAC-colorings

If a valuation yields two active NAC-colorings δ, δ', then the set $\{(\delta(e), \delta'(e)) \colon e \in E_G\}$ has 3 elements.



Orthogonal diagonals



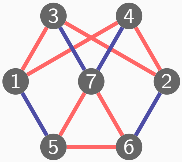
Lemma (GLS)

If there is an active NAC-coloring δ *of an algebraic motion of* (G, λ) *such that a 4-cycle* $(1, 2, 3, 4)$ *is blue and there are red paths from* 1 *to* 3 *and from* 2 *to* 4*, then*

$$\lambda_{12}^2 + \lambda_{34}^2 = \lambda_{23}^2 + \lambda_{14}^2 \,,$$

namely, the 4-cycle $(1, 2, 3, 4)$ *has orthogonal diagonals.*

Triangle in Q_1



$$\implies \lambda_{57}^2 r^2 + \lambda_{67}^2 s^2 + \left(\lambda_{56}^2 - \lambda_{57}^2 - \lambda_{67}^2\right) rs = 0\,,$$

$$r = \lambda_{24}^2 - \lambda_{23}^2,\ s = \lambda_{14}^2 - \lambda_{13}^2$$

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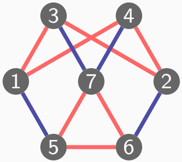

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Considering the equation as a polynomial in r, the discriminant is

$$(\lambda_{56}+\lambda_{57}+\lambda_{67})(\lambda_{56}+\lambda_{57}-\lambda_{67})(\lambda_{56}-\lambda_{57}+\lambda_{67})(\lambda_{56}-\lambda_{57}-\lambda_{67})s^2\,.$$

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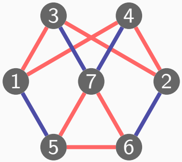

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Theorem (GLS)

The vertices 5, 6 and 7 are collinear for every proper flexible labeling of Q_1.

Ramification formula

Theorem (GLS)

Let $\mathcal{C}$ be an algebraic motion of (G,λ) with the set of active NAC-colorings N. There exist $\mu_\delta \in \mathbb{Z}_{\geq 0}$ for all NAC-colorings δ of G such that:

1. $\mu_\delta \neq 0$ *if and only if* $\delta \in N$, *and*
2. *for every 4-cycle* (V_i, E_i) *of* G, *there exists a positive integer* d_i *such that*

$$\sum_{\substack{\delta \in \mathsf{NAC}_G \\ \delta|_{E_i} = \delta'}} \mu_\delta = d_i \qquad \text{for all NAC-colorings } \delta' \in \{\delta|_{E_i} : \delta \in N\}.$$

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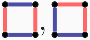
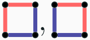
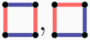
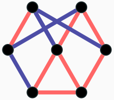
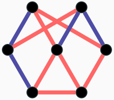
$\mathfrak{p} = \{$  $\}$, $\mathfrak{o} = \{$  $\}$, $\mathfrak{g} = \{$  $\}$,

$\mathfrak{a} = \{$  $\}$, $\mathfrak{e} = \{$  $\}$.

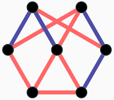
Example



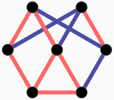
ϵ_{13}



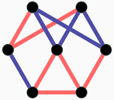
ϵ_{14}



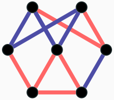
ϵ_{23}



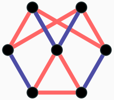
ϵ_{24}



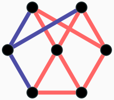
γ_1



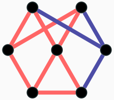
γ_2



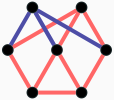
η



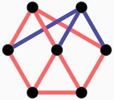
ψ_1



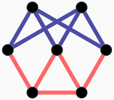
ψ_2



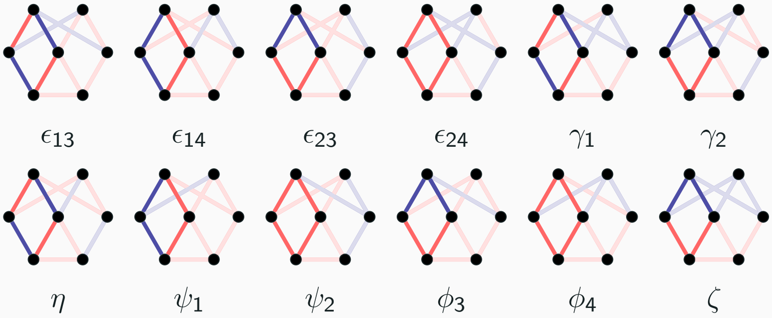
ϕ_3

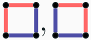


ϕ_4

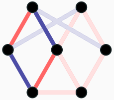


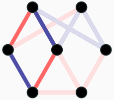
ζ



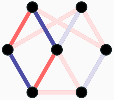
Antiparallelogram () $\Longrightarrow$

Example

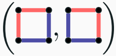




ϵ_{13} ϵ_{14} ϵ_{23} ϵ_{24} γ_1 γ_2

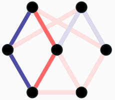


η ψ_1 ψ_2 ϕ_3 ϕ_4 ζ

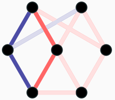
Antiparallelogram  $\implies$

$$\mu_{\epsilon_{13}} = \mu_{\gamma_1} = \mu_\eta = 0$$

Example



ϵ_{13} ϵ_{14} ϵ_{23} ϵ_{24} γ_1 γ_2



η ψ_1 ψ_2 ϕ_3 ϕ_4 ζ

Antiparallelogram ( , ) $\Longrightarrow$

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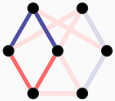
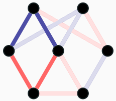
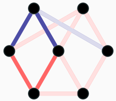
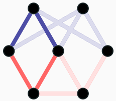
$$\mu_{\epsilon_{14}} + \mu_{\psi_1}$$

ϵ_{13} ϵ_{14} ϵ_{23} ϵ_{24} γ_1 γ_2

η ψ_1 ψ_2 ϕ_3 ϕ_4 ζ

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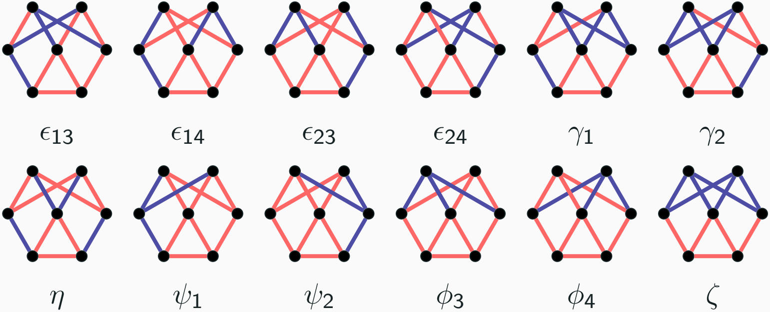

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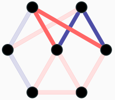
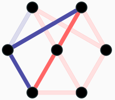
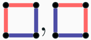
Example



ϵ_{13} ϵ_{14} ϵ_{23} ϵ_{24} γ_1 γ_2



η ψ_1 ψ_2 ϕ_3 ϕ_4 ζ

Antiparallelogram () $\Longrightarrow$

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Classification of motions

- Find all possible types of motions of quadrilaterals with consistent μ_δ's

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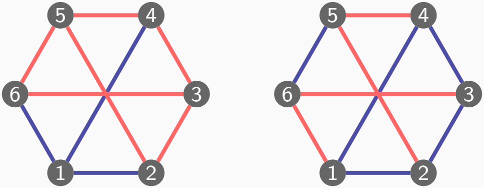
Implementation – SageMath package `FlexRiLoG` (`https://github.com/Legersky/flexrilog`)

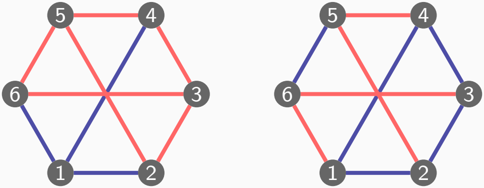
$K_{3,3}$



ω_1 ϵ_{56}

- Find consistent motion types of 4-cycles – 112 out of
  - $32\,768 = 2^{15}$ subsets of NAC-colorings, or
  - $1\,953\,125 = 5^9$ motion types of 4-cycles, or
  - $3\,075$ motion types of 4-cycles iteratively

$K_{3,3}$



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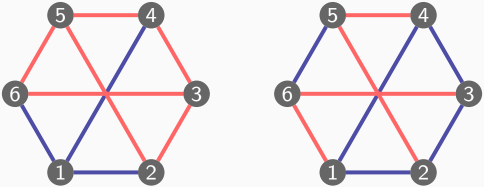
$K_{3,3}$



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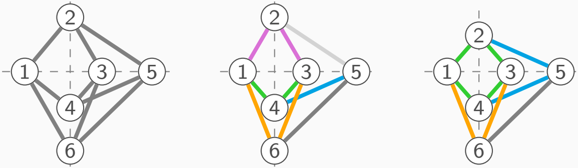
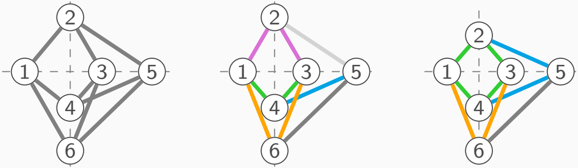
- Identify symmetric cases – 4 classes

Classification of motions of $K_{3,3}$



4-cycles	active NAC-colorings	#	
$\mathfrak{ggggggggg}$	$\mathrm{NAC}_{K_{3,3}}$	1	
$\mathfrak{oooggggggg}$	$\{\epsilon_{12}, \epsilon_{23}, \epsilon_{34}, \epsilon_{14}, \epsilon_{16}, \epsilon_{36}, \omega_1, \omega_3\}$	6	Dixon I
$\mathfrak{pooggogge}$	$\{\epsilon_{12}, \epsilon_{23}, \epsilon_{34}, \epsilon_{14}\}$	9	
$\mathfrak{pgggaggag}$	$\{\epsilon_{12}, \epsilon_{34}, \omega_5, \omega_6\}$	18	Dixon II

Classification of motions of $K_{3,3}$



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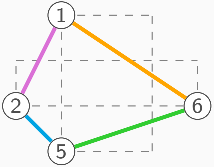

$(1, 2, 5, 6)$ – perpendicular diagonals

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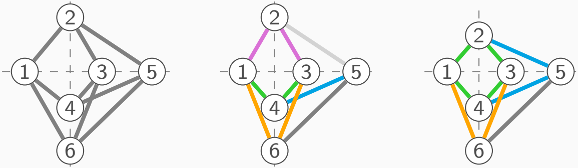

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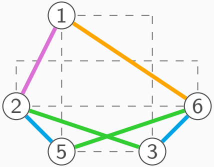
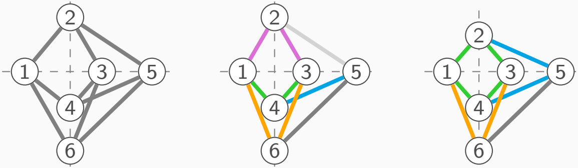


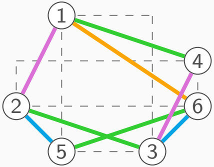
$(1, 2, 5, 6)$ – perpendicular diagonals

$(2, 3, 6, 5)$ – antiparallelogram

Classification of motions of $K_{3,3}$



4-cycles	active NAC-colorings	#	
ggggggggg	$\mathrm{NAC}_{K_{3,3}}$	1	
ooogggggg	$\{\epsilon_{12}, \epsilon_{23}, \epsilon_{34}, \epsilon_{14}, \epsilon_{16}, \epsilon_{36}, \omega_1, \omega_3\}$	6	Dixon I
pooggogge	$\{\epsilon_{12}, \epsilon_{23}, \epsilon_{34}, \epsilon_{14}\}$	9	
pgggaggag	$\{\epsilon_{12}, \epsilon_{34}, \omega_5, \omega_6\}$	18	Dixon II

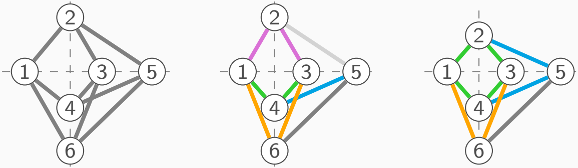


$(1, 2, 5, 6)$ – perpendicular diagonals

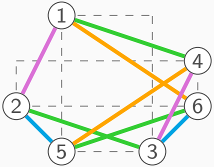
$(2, 3, 6, 5)$ – antiparallelogram

$(1, 2, 3, 4)$ – parallelogram

Classification of motions of $K_{3,3}$



4-cycles	active NAC-colorings	#	
$\mathfrak{ggggggggg}$	$\mathrm{NAC}_{K_{3,3}}$	1	
$\mathfrak{ooogggggg}$	$\{\epsilon_{12}, \epsilon_{23}, \epsilon_{34}, \epsilon_{14}, \epsilon_{16}, \epsilon_{36}, \omega_1, \omega_3\}$	6	Dixon I
$\mathfrak{pooggogge}$	$\{\epsilon_{12}, \epsilon_{23}, \epsilon_{34}, \epsilon_{14}\}$	9	
$\mathfrak{pgggaggag}$	$\{\epsilon_{12}, \epsilon_{34}, \omega_5, \omega_6\}$	18	Dixon II



$(1, 2, 5, 6)$ – perpendicular diagonals

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$(1, 2, 3, 4)$ – parallelogram

$(1, 4, 5, 6)$ – antiparallelogram

Classification of motions of Q_1

4-cycles	active NAC-colorings	#	type	dim.
$\mathfrak{pggpgpg}$	$\{\epsilon_{13}, \epsilon_{24}, \eta\}$	2	I	4
$\mathfrak{poapope}$	$\{\epsilon_{13}, \eta\}$	4	$\subset$ I, IV$_-$, V, VI	2
$\mathfrak{peepapa}$	$\{\epsilon_{13}, \epsilon_{24}\}$	2	$\subset$ I, II, III	2
$\mathfrak{ogggggg}$	$\{\epsilon_{ij}, \gamma_1, \gamma_2, \psi_1, \psi_2\}$	1	II$_- \cup$ II$_+$	5
$\mathfrak{peegggg}$	$\{\epsilon_{13}, \epsilon_{14}, \epsilon_{23}, \epsilon_{24}\}$	1	$\subset$ II$_-$, II$_+$	4
$\mathfrak{oggpgga}$	$\{\epsilon_{13}, \epsilon_{24}, \gamma_1, \psi_2\}$	4	$\subset$ II$_-$	3
$\mathfrak{oggegge}$	$\{\epsilon_{13}, \epsilon_{23}, \gamma_1, \gamma_2\}$	2	$\subset$ II$_-$, deg.	2
$\mathfrak{ogggaga}$	$\{\epsilon_{13}, \epsilon_{24}, \psi_1, \psi_2, \zeta\}$	2	III	3
$\mathfrak{ggapggg}$	$\{\epsilon_{13}, \eta, \phi_4, \psi_2\}$	4	IV$_- \cup$ IV$_+$	4
$\mathfrak{ggaegpe}$	$\{\epsilon_{13}, \eta, \gamma_2, \phi_3\}$	4	V	3
$\mathfrak{pggegge}$	$\{\epsilon_{13}, \epsilon_{23}, \eta, \zeta\}$	2	VI	3

μ-numbers

Let $\mathcal{C}$ be an algebraic motion. For $\delta \in \mathsf{NAC}_G$ and a valuation ν:

$$\mathrm{gap}(\delta, \nu) := \max \left\{ 0, \min_{\substack{e \in E_G \\ \delta(e) = \mathrm{red}}} \nu(W_e) - \max_{\substack{e \in E_G \\ \delta(e) = \mathrm{blue}}} \nu(W_e) \right\}$$

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and

$$\mu(\delta,\mathcal{C}) := \sum_{\nu\in\mathsf{Val}(\mathcal{C})} \mathsf{gap}(\delta,\nu)\,.$$

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and

$$\mu(\delta, \mathcal{C}) := \sum_{\nu \in \mathsf{Val}(\mathcal{C})} \mathsf{gap}(\delta, \nu)\,.$$

The set of active NAC-colorings satisfies

$$\mathsf{NAC}_G(\mathcal{C}) = \{\delta \in \mathsf{NAC}_G : \mu(\delta, \mathcal{C}) \neq 0\}\,.$$

Ramification formula

Theorem (GLS)

Let $\mathcal{C}$ be an algebraic motion of (G, λ). Let G' be a subgraph of G and $f : \mathcal{C} \to \mathcal{C}'$ be the projection of $\mathcal{C}$ into realizations of G', where $\mathcal{C}'$ is an algebraic motion of G'. If δ' be a NAC-coloring of G', then

$$\sum_{\substack{\delta \in \mathrm{NAC}_G \\ \delta|_{E_{G'}} = \delta'}} \mu(\delta, \mathcal{C}) = \mu(\delta', \mathcal{C}') \cdot \deg(f)\,.$$

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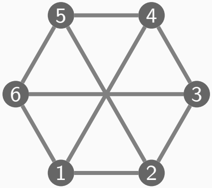
Theorem (GLS)

Let C_4 be a 4-cycle graph with an algebraic motion $\mathcal{C}'$. If δ' is an active NAC-coloring of $\mathcal{C}'$, then $\mu(\delta', \mathcal{C}') = 1$.

Computer-free proof for $K_{3,3}$

Let $\mathcal{C}'$ be an algebraic motion of $K_{3,3}$. Let p_i and p_{ij} be the projections removing vertices, i and i, j (different parity).



Proposition (Gallet, GLS)

If $\mathcal{C}'$ is not a Dixon I, then all the maps $p_1, \ldots, p_6$ are birational.

Computer-free proof for $K_{3,3}$ – Dixon I excluded

There are only 26 *degree tables* modulo symmetries:

$$\begin{array}{cccc} & 2 & 4 & 6 \\ 1 & \ddots & \vdots & \iddots \\ 3 & \cdots & \deg p_{ij} \in \{1,2\} & \cdots \\ 5 & \iddots & \vdots & \ddots \end{array}$$

Computer-free proof for $K_{3,3}$ – Dixon I excluded

There are only 26 *degree tables* modulo symmetries:

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Each table determines motion types of all 4-cycles (almost), not all of them possible:

$$\begin{array}{ccc} \mathfrak{g} & \mathfrak{g} & \mathfrak{g} \\ \mathfrak{g} & \mathfrak{g} & \mathfrak{g} \\ \mathfrak{g} & \mathfrak{g} & \mathfrak{g} \end{array} \qquad \begin{array}{ccc} \mathfrak{a} & \mathfrak{p} & \mathfrak{e} \\ \mathfrak{p} & \mathfrak{a} & \mathfrak{e} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{r} \end{array} \qquad \begin{array}{ccc} \mathfrak{p}/\mathfrak{a}/\mathfrak{r} & \mathfrak{g} & \mathfrak{g} \\ \mathfrak{g} & \mathfrak{p}/\mathfrak{a}/\mathfrak{r} & \mathfrak{g} \\ \mathfrak{g} & \mathfrak{g} & \mathfrak{p}/\mathfrak{a}/\mathfrak{r} \end{array}$$

Computer-free proof for $K_{3,3}$ – Dixon I excluded

There are only 26 *degree tables* modulo symmetries:

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Each table determines motion types of all 4-cycles (almost), not all of them possible:

$$\begin{array}{ccc} \mathfrak{a} & \mathfrak{g} & \mathfrak{g} \\ \mathfrak{g} & \mathfrak{a} & \mathfrak{g} \\ \mathfrak{g} & \mathfrak{g} & \mathfrak{p} \end{array}$$

Thank you

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