Method for construction of parallel addition algorithms

Jan Legerský, Milena Svobodová

FNSPE, CTU in Prague
jan.legersky@gmail.com

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Preliminaries

Positional numeration system $(\beta, \mathcal{A})$ is defined by

- *Base* $\beta \in \mathbb{C}, |\beta| > 1$.
- Finite *digit set* $\mathcal{A} \subset \mathbb{Z}$ containing 0, usually called *alphabet*.

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A complex number x has a finite $(\beta, \mathcal{A})$-representation if $x = \sum_{j=-m}^{n} x_j \beta^j$ with coefficients x_j in $\mathcal{A}$.

$$x = x_n x_{n-1} \cdots x_1 x_0 \bullet x_{-1} x_{-2} \cdots x_{-m}$$

Addition

$$
\begin{array}{rl}
x = x_n x_{n-1} \cdots x_1 x_0 \bullet x_{-1} x_{-2} \cdots x_{-m} & , x_i \in \mathcal{A} \\
+ & \\
y = y_n y_{n-1} \cdots y_1 y_0 \bullet y_{-1} y_{-2} \cdots y_{-m} & , y_i \in \mathcal{A} \\
\hline
w = w_n w_{n-1} \cdots w_1 w_0 \bullet w_{-1} w_{-2} \cdots w_{-m} \, , &
\end{array}
$$

where

$$w_j = x_j + y_j \in \mathcal{A} + \mathcal{A} \, .$$

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Goal – find $(\beta, \mathcal{A})$-representation of w, i.e. a sequence

$$z_{n'} z_{n'-1} \cdots z_1 z_0 z_{-1} z_{-2} \cdots z_{-m'}$$

such that $z_j \in \mathcal{A}$ and

$$z_{n'} z_{n'-1} \cdots z_1 z_0 \bullet z_{-1} z_{-2} \cdots z_{-m'} = w$$

We have $0 = 1 \cdot \beta^j - \beta \cdot \beta^{j-1} = 1(-\beta)0 \cdots 0\bullet$
$\rightarrow$ We add $q_j \cdot 0$ for each j:
$0 = q_j \cdot \beta^j - \beta q_j \cdot \beta^{j-1}$

$$
\begin{array}{ccccccc}
w_n w_{n-1} & \cdots & w_{j+1} & w_j & w_{j-1} & \cdots & w_1 w_0\bullet \\
 & & & & q_{j-2} & \cdot^{\cdot^{\cdot}} & \\
 & & & q_{j-1} & -\beta q_{j-1} & & \\
 & & q_j & -\beta q_j & & & \\
 & \cdot^{\cdot^{\cdot}} & -\beta q_{j+1} & & & & \\
\hline
z_{n+1} z_n z_{n-1} & \cdots & z_{j+1} & z_j & z_{j-1} & \cdots & z_1\ z_0\bullet
\end{array}
$$

$\Longrightarrow$

$$z_j = w_j + q_{j-1} - q_j\beta \in \mathcal{A}$$

Assume now a standard numeration system $(\beta, \mathcal{A})$:

$$\beta \in \mathbb{N}\,, \beta \geq 2\,, \mathcal{A} = \{0, 1, 2, \ldots, \beta - 1\}$$

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Conversion runs from right to left:

$$\begin{array}{ll} w_n w_{n-1} \cdots w_{j+1} w_j w_{j-1} \cdots w_1 w_0 \bullet & , w_i \in \mathcal{A} + \mathcal{A}, \\ \longrightarrow z_{n+1}\, z_n\, z_{n-1}\, \cdots z_{j+1} z_j\, z_{j-1}\, \cdots\, z_1\, z_0 \bullet & , z_i \in \mathcal{A}. \end{array}$$

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$$z_j = w_j + q_{j-1} - q_j \beta$$

The weight coefficients q_j and digits z_j are unique in the standard numeration system

$$\Longrightarrow\ z_j = z_j(w_j, \ldots, w1, w_0)$$

Thus, addition is linear in length of inputs.

Parallel Addition

Introduced by Avizienis in 1961:

$$\cdots w_{j+t+1} w_{j+t} \cdots w_{j+1} w_j w_{j-1} \cdots w_{j-r} w_{j-r-1} \cdots \quad , w_i \in \mathcal{A} + \mathcal{A},$$
$$\longrightarrow \cdots z_{j+t+1}\ z_{j+t}\ \cdots\ z_{j+1} z_j\ z_{j-1}\ \cdots z_{j-r}\ z_{j-r-1}\ \cdots \quad , z_i \in \mathcal{A}.$$

Digit conversion is a mapping $(\mathcal{A} + \mathcal{A})^{t+r+1} \to \mathcal{A}$:

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Thus the parallel addition is done in constant time but the numeration system must be redundant – a number has more than one admissible representation.
For example:

$$\beta \in \mathbb{N}, \beta \geq 3, \mathcal{A} = \{-a, \ldots, 0, \ldots a\}, b/2 < a \leq b - 1 .$$

Non-standard numeration systems

Integer alphabets:

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- Base $\beta \in \mathbb{Z}[\omega] = \left\{ \sum_{j=0}^{d-1} a_j \omega^j \ : a_j \in \mathbb{Z} \right\}$, where $\omega \in \mathbb{C}$ is an algebraic integer of degree d.

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- Only few manually found algorithms.

How do we find systematically algorithms with minimal alphabets?

Method for construction of parallel addition algorithms

Key problem – find weight coefficients q_j such that

$$z_j = \underbrace{w_j}_{\in \mathcal{A}+\mathcal{A}} + q_{j-1} - q_j\beta \in \mathcal{A}$$

for any input w and every j.

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In order to do that we need to find $M \in \mathbb{N}$ and
$q : (\mathcal{A}+\mathcal{A})^M \to \mathcal{Q} \subset \mathbb{Z}[\omega]$ such that $q_j = q(w_j, \ldots, w_{j-M+1})$.
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Our method:

1. Find set $\mathcal{Q} \subset \mathbb{Z}[\omega]$ of possible weight coefficients.
2. Increment M and for each $w_j, w_{j-1}, \ldots, w_{j-M+1} \in (\mathcal{A}+\mathcal{A})^M$ try to find a weight coefficient from $\mathcal{Q}$ to define q.

Phase 1 – searching for the set of weight coefficients

We want to find set of weight coefficients $\mathcal{Q} \subset \mathbb{Z}[\omega]$ such that

$$\underbrace{(\mathcal{A} + \mathcal{A})} + \underbrace{\mathcal{Q}} \subset \underbrace{\mathcal{A}} + \underbrace{\beta\mathcal{Q}}$$

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It implies that there is $q_j \in \mathcal{Q}$ such that

$$z_j = w_j + q_{j-1} - q_j\beta \in \mathcal{A}\,.$$

for all $q_{j-1} \in \mathcal{Q}$ and $w_j \in \mathcal{A}+\mathcal{A}$.

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Phase 1

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Repeat:

- extend $\mathcal{Q}_k$ to $\mathcal{Q}_{k+1}$ in a minimal possible way so that

$$(\mathcal{A} + \mathcal{A}) + \mathcal{Q}_k \subset \mathcal{A} + \beta \mathcal{Q}_{k+1},$$

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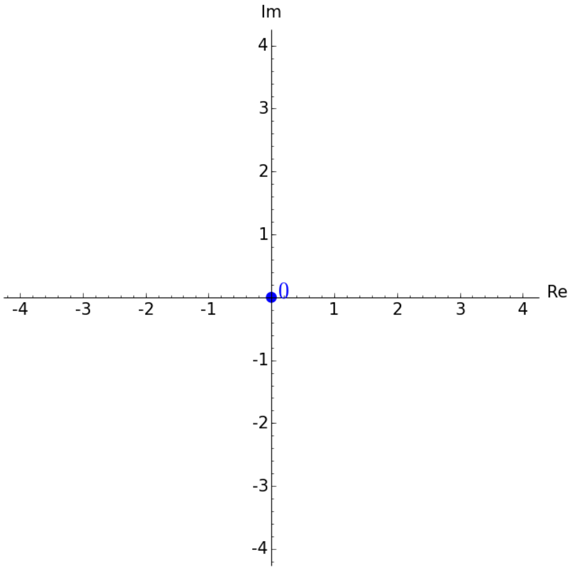
$\mathcal{Q} := \mathcal{Q}_k$

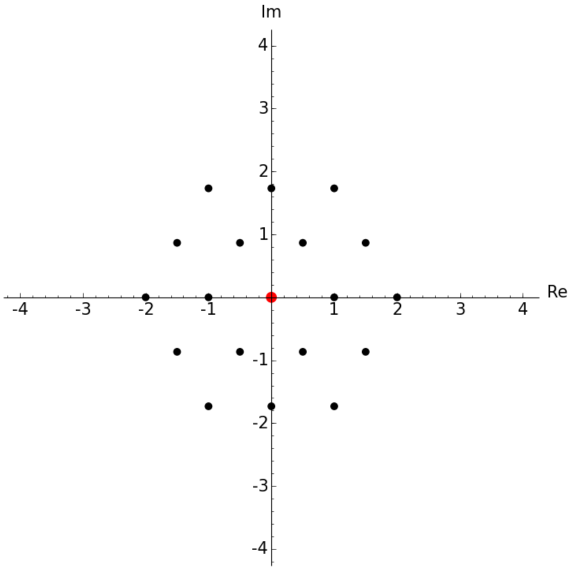
Example - phase 1

Eisenstein numeration system

- Base $\beta = \omega - 1$, where $\omega = \exp(\frac{2\pi i}{3}), \omega^2 + \omega + 1 = 0$.
- Minimal polynomial of the base is $\beta^2 + 3\beta + 3$.
- Alphabet $\mathcal{A} = \{0, 1, -1, \omega, -\omega, -\omega - 1, \omega + 1\} \subset \mathbb{Z}[\omega]$.

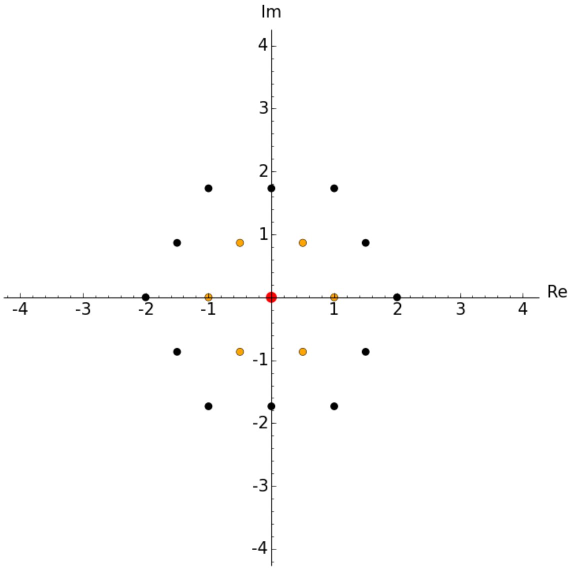


$\mathcal{Q}_0$



$\mathcal{Q}_0$

$\mathcal{A} + \mathcal{A} + \mathcal{Q}_0$

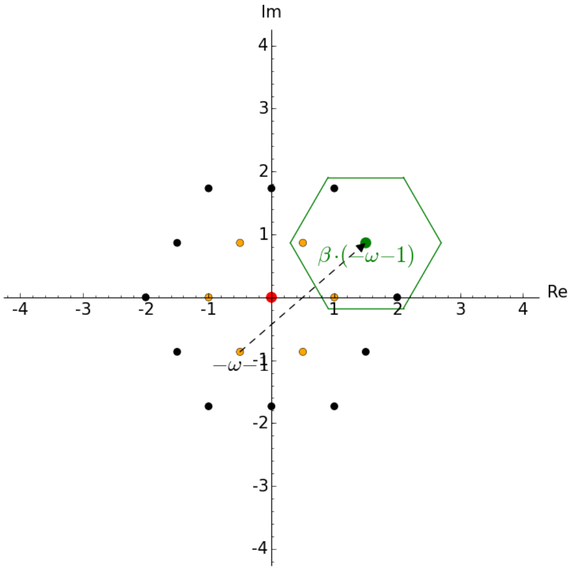


$\mathcal{Q}_0$

$\mathcal{A}+\mathcal{A}+\mathcal{Q}_0$

$? \subset ?$

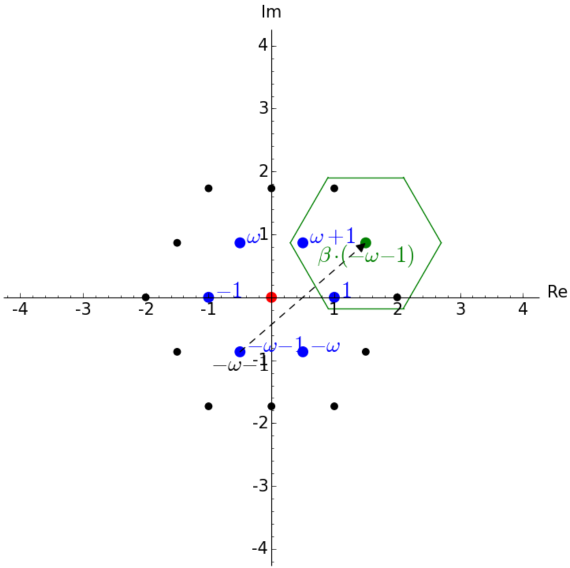
$\mathcal{A}+\beta\cdot\mathcal{Q}_0$



$\mathcal{Q}_0$

$\mathcal{A}+\mathcal{A}+\mathcal{Q}_0$

$\mathcal{A}+\beta\cdot(-\omega-1)$



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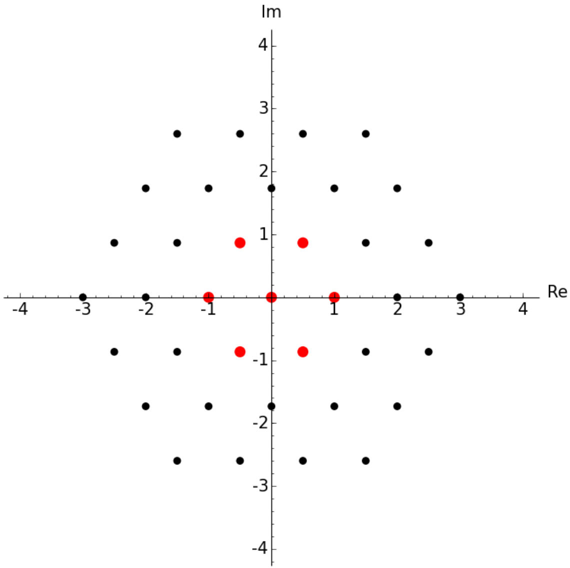
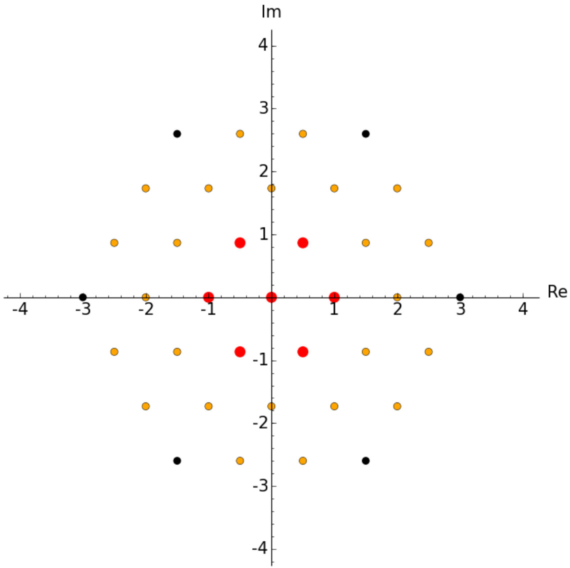
$\mathcal{Q}_1\setminus\mathcal{Q}_0$



$\mathcal{Q}_1$

$\mathcal{A} + \mathcal{A} + \mathcal{Q}_1$



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$? \subset ?$

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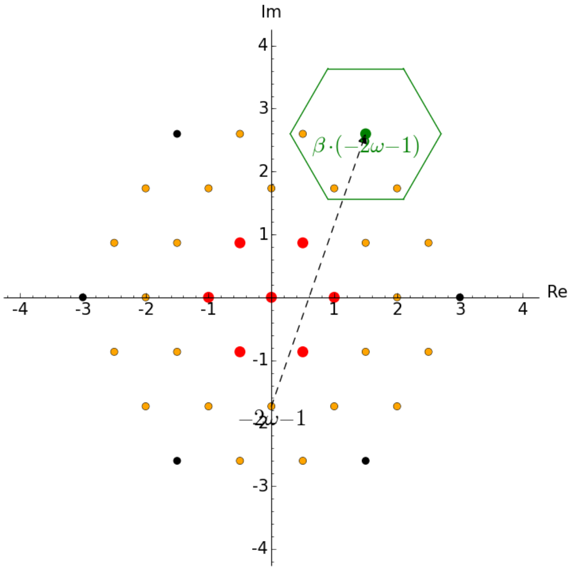
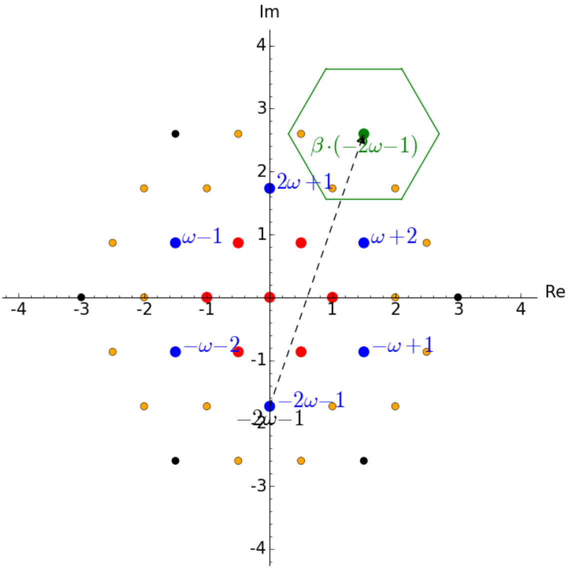


$\mathcal{Q}_1$

$\mathcal{A}+\mathcal{A}+\mathcal{Q}_1$

$\mathcal{A}+\beta\cdot(-2\omega-1)$



$\mathcal{Q}_1$

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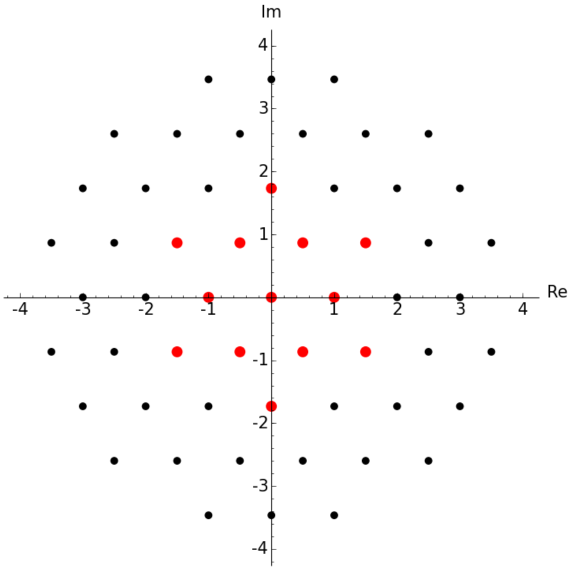
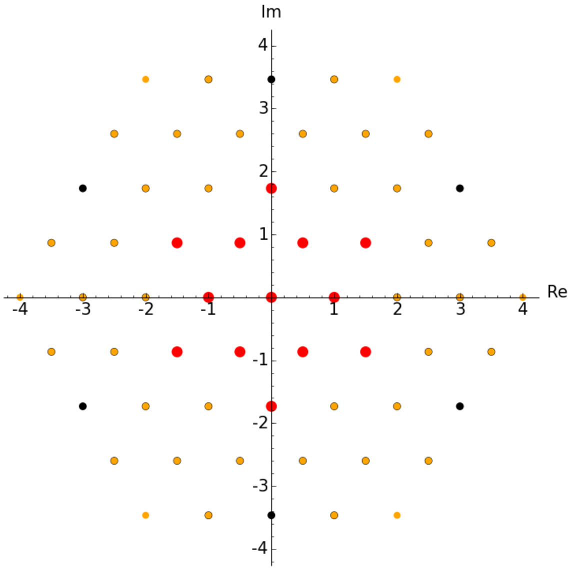
$\mathcal{A}+\beta\cdot(-2\omega-1)$

$\mathcal{Q}_2\backslash\mathcal{Q}_1$



$\mathcal{Q}_2$

$\mathcal{A} + \mathcal{A} + \mathcal{Q}_2$



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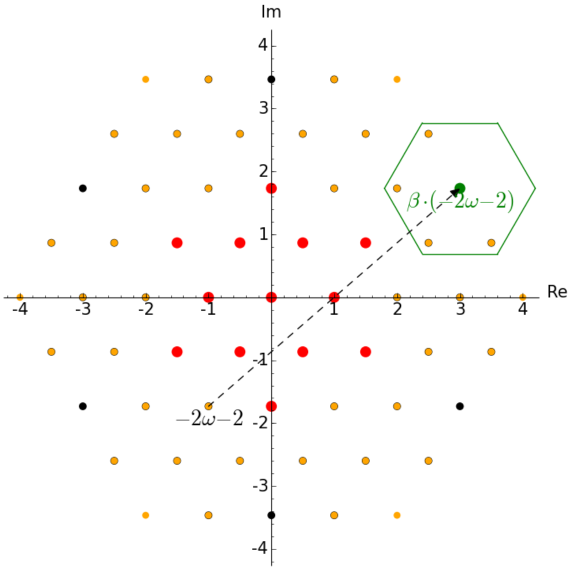

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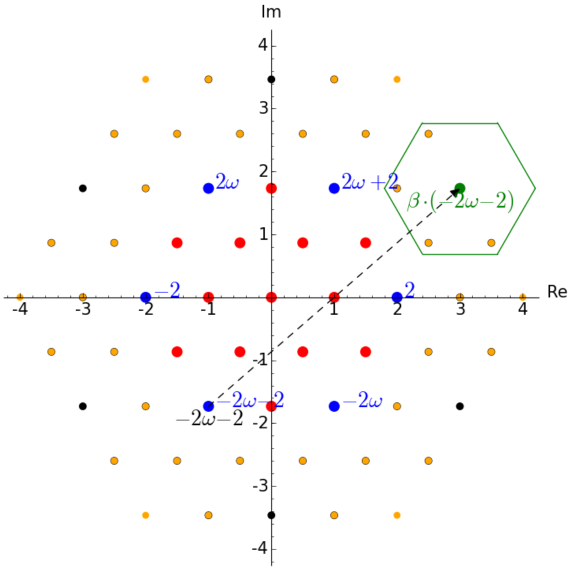

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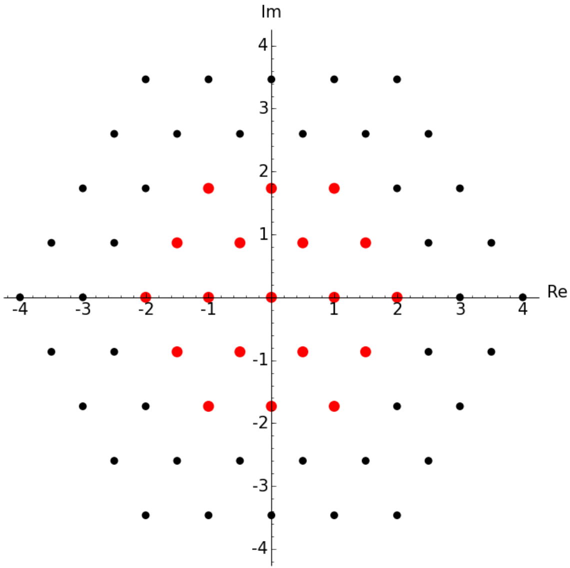

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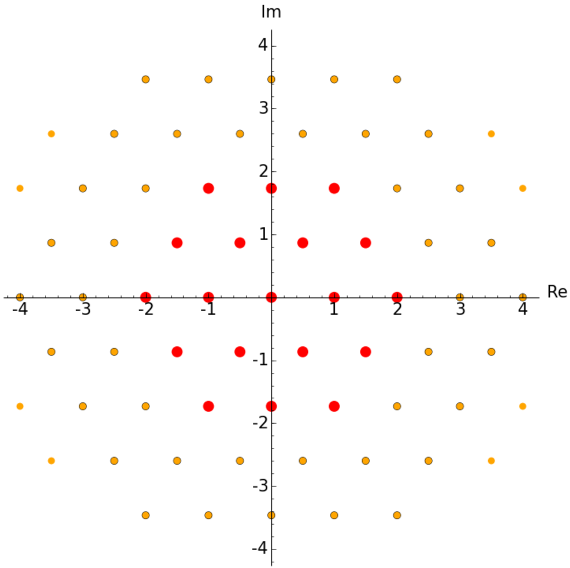
$\mathcal{A} + \mathcal{A} + \mathcal{Q}_3$



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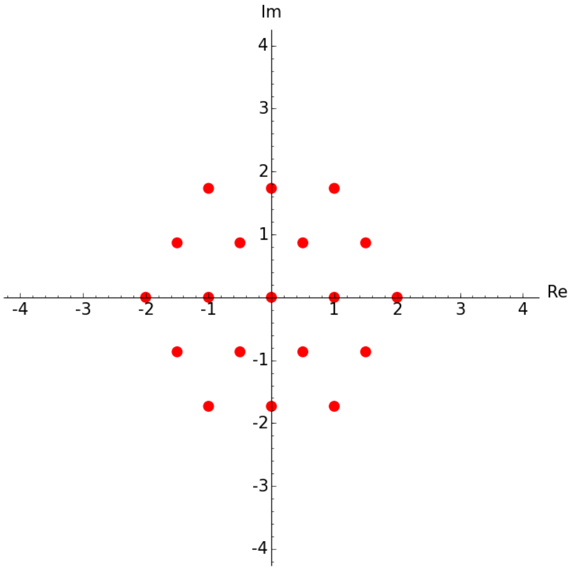

$\mathcal{Q} = \mathcal{Q}_3$

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The set of all such needed values of q_j is denoted by $\mathcal{Q}_{[w_j,\ldots,w_{j-m+1}]} \subset \mathcal{Q}$

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The length M and weight function q is found when

$$\#\mathcal{Q}_{[w_j,\dots,w_{j-M+1}]} = 1$$

for all $w_j, \dots, w_{j-M+1} \in (\mathcal{A}+\mathcal{A})^M$.

Phase 2

$m := 1$

For each $w_j \in \mathcal{A} + \mathcal{A}$ find minimal set $\mathcal{Q}_{[w_j]} \subset \mathcal{Q}$ such that

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While $(\max\{\#\mathcal{Q}_{[w_j,\dots,w_{j-m+1}]} : (w_j,\dots,w_{j-m+1}) \in (\mathcal{A}+\mathcal{A})^m\} > 1)$ do:

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While $(\max\{\#\mathcal{Q}_{[w_j,\ldots,w_{j-m+1}]} : (w_j, \ldots, w_{j-m+1}) \in (\mathcal{A}+\mathcal{A})^m\} > 1)$ do:

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$q(w_j,\dots,w_{j-M+1}) :=$ only element of $\mathcal{Q}_{[w_j,\dots,w_{j-M+1}]}$

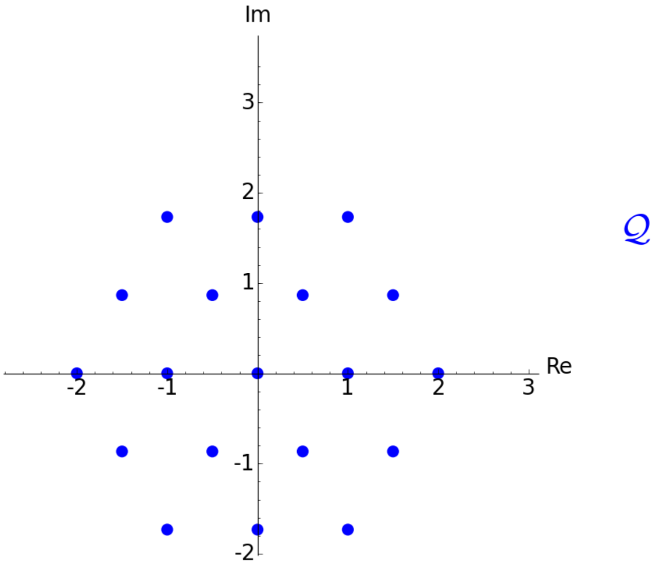
Now we have parallel conversion algorithm:

$$
\begin{aligned}
z_j &= w_j + q_{j-1} - q_j\beta = \\
&= w_j + q(w_{j-1}, w_{j-2}, \dots, w_{j-M}) - \beta q(w_j, w_{j-1}, \dots, w_{j-M+1}) = \\
&= z_j(w_j, w_{j-1}, \dots, w_{j-M})\,.
\end{aligned}
$$

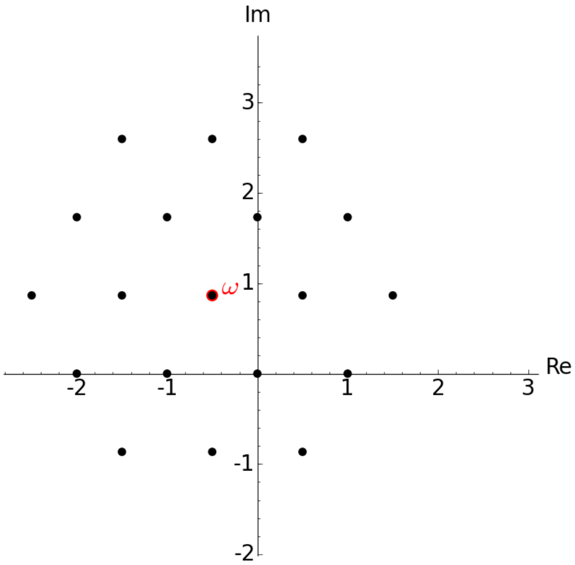
Example - phase 2

Let us continue with Eisenstein numeration system.
Assume the input is $w = \omega\,1\,2$.

Input: $w = \omega\, 1\, 2$

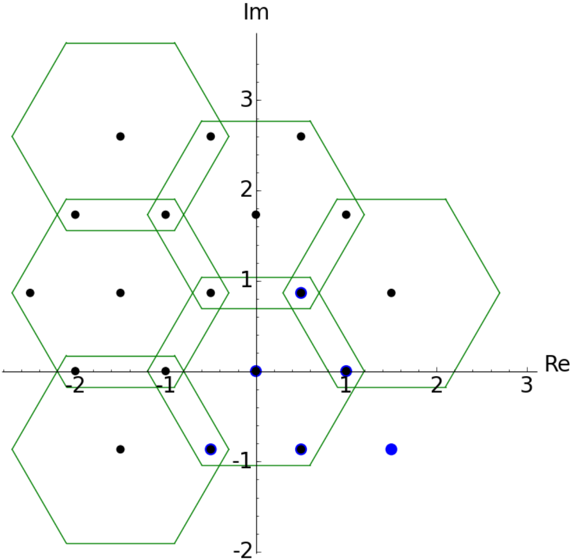


Input: $w = \omega\, 1\, 2$



$\omega + \mathcal{Q}$

Input: $w = \omega\, 1\, 2$



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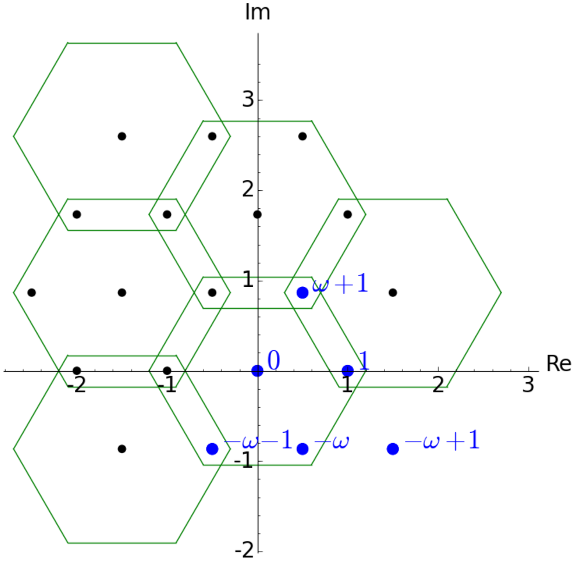
$? \subset ?$

$\mathcal{A} + \beta \cdot \mathcal{Q}_{[\omega]}$

Input: $w = \omega\, 1\, 2$



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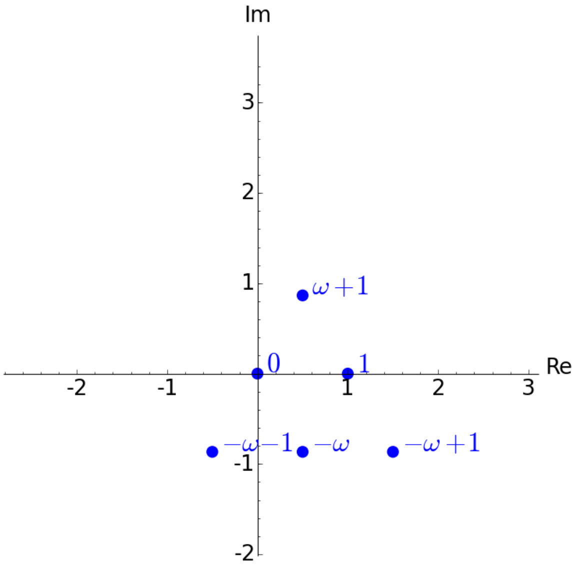

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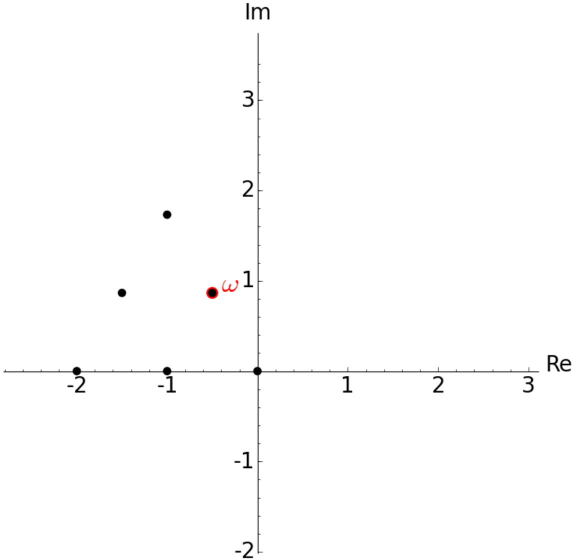

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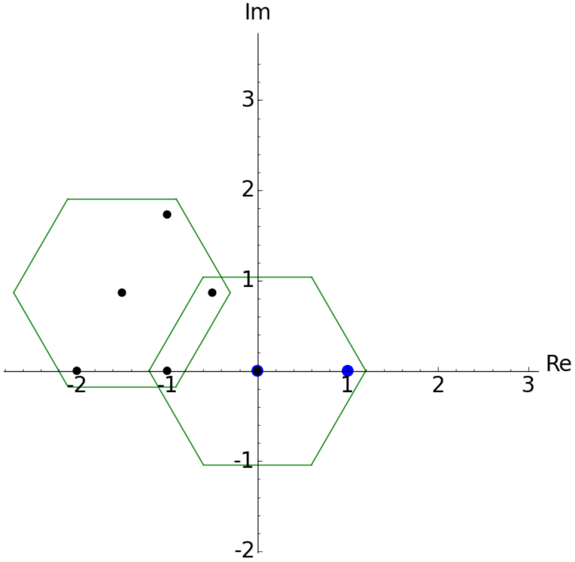

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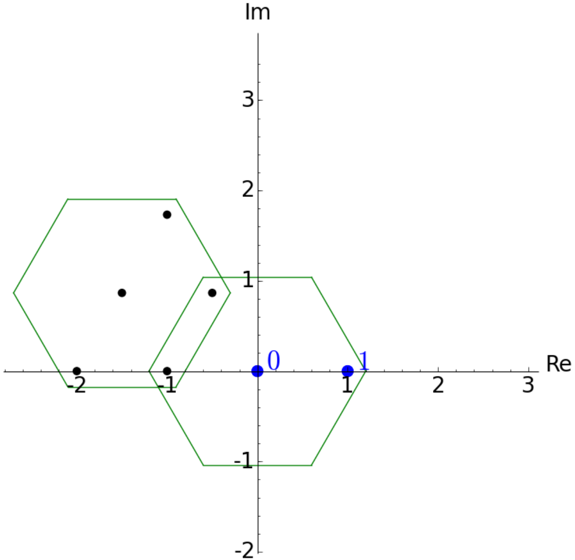

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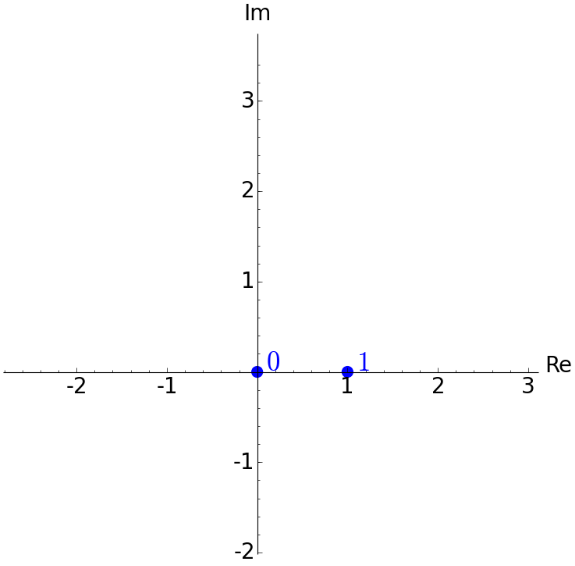

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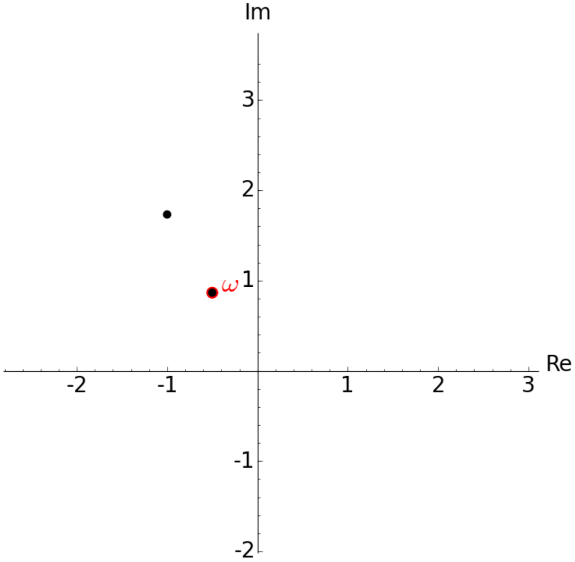

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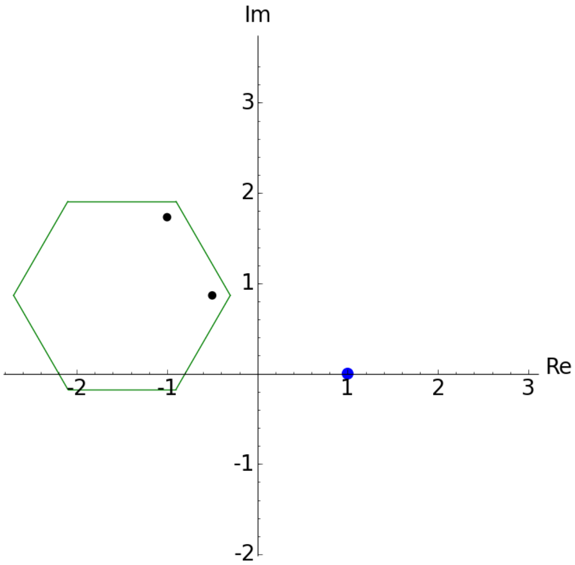

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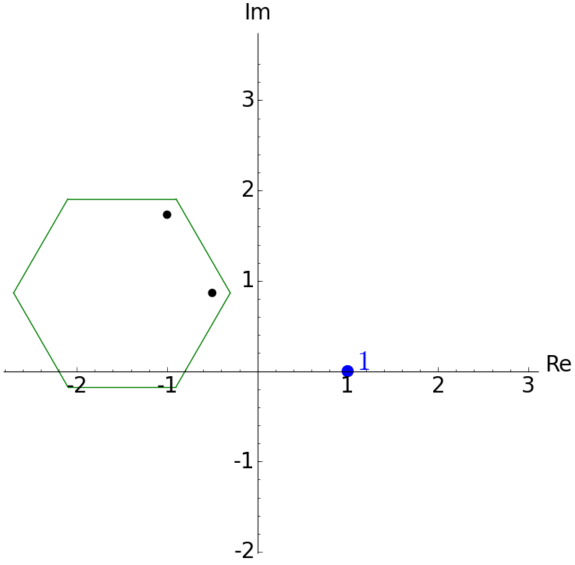

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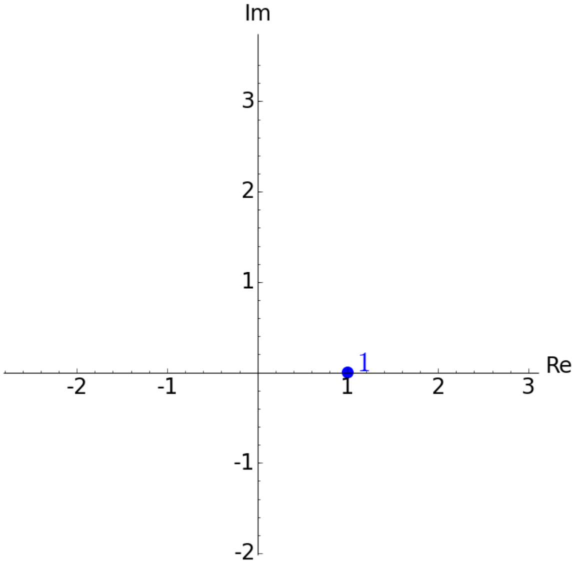

$\mathcal{Q}_{[\omega,1,2]}$
$= q(\omega,1,2)$

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- Experimental implementation in Sage gives good results for several tested cases.

Result examples

Successful: parallel addition algorithms found by the program

- Eisenstein: $\beta^2 + 3\beta + 3 = 0$ with $\mathcal{A} \subset \mathbb{Z}[\beta]$ of size $\#\mathcal{A} = 7$
- Penney: $\beta^2 + 2\beta + 2 = 0$ with $\mathcal{A} \subset \mathbb{Z}[\beta]$ of size $\#\mathcal{A} = 5$
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- quadratic real: $\beta^2 - 5\beta + 3 = 0$ with $\mathcal{A} \subset \mathbb{Z}$ of size $\#\mathcal{A} = 7$
  (failure / non-convergence already in Phase 1.)

Open Issues & Ideas

What are the mathematical reasons for (non)convergence of the iterative method ?

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- Use a different metric in $\mathbb{C}$ than the 'classical' one to guarantee convergence of Phase 1 for a broader class of bases.

Thank you