Graphs with flexible labelings

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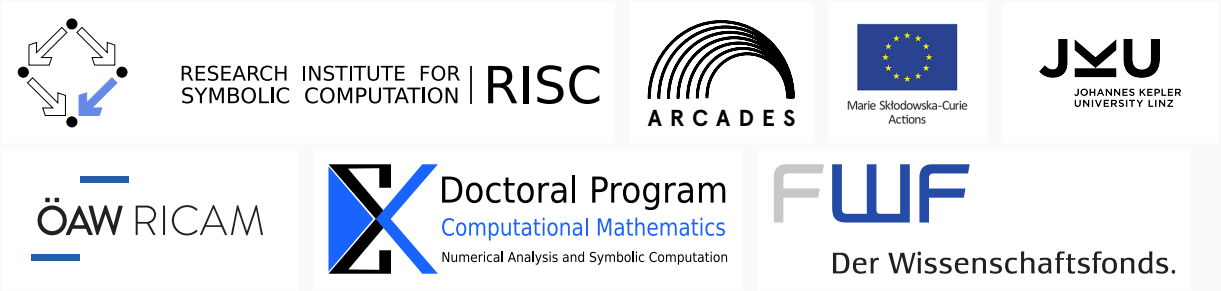
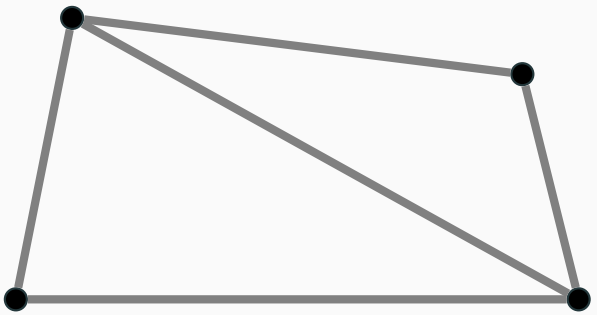
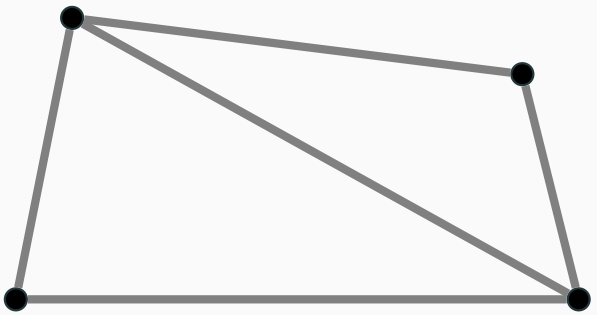
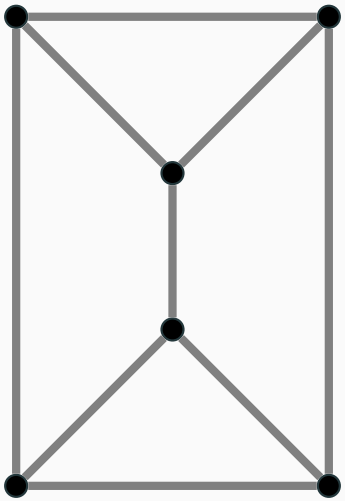
RISC JKU Linz, Austria

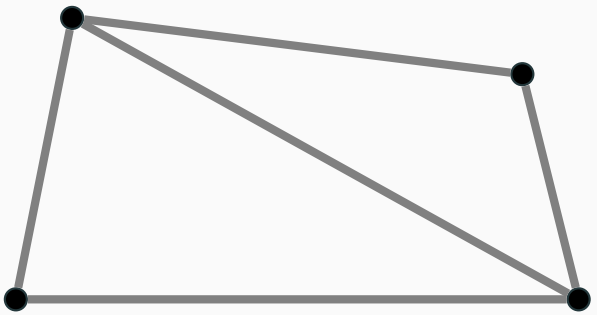
Bond-node structures: rigidity, combinatorics and chemistry
Lancaster, June 6, 2018

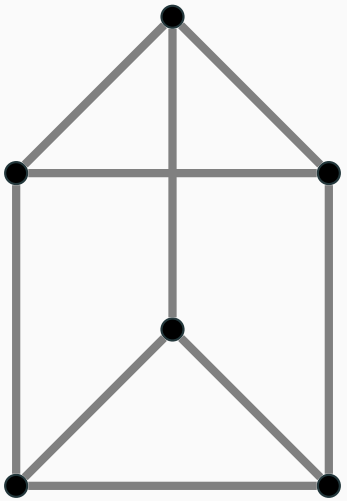


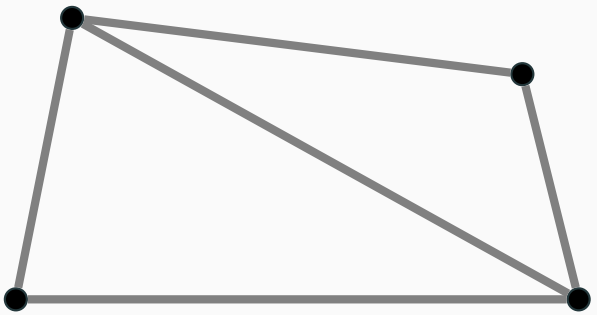


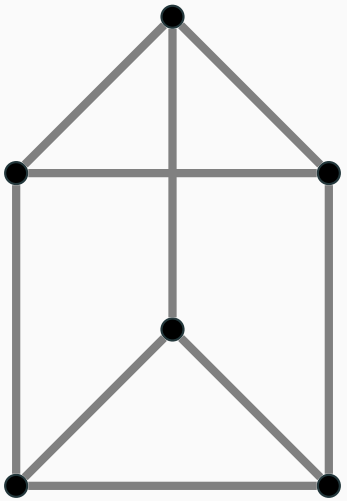












Flexible labelings of graphs

Definition

Let $G = (V, E)$ be a simple connected graph.

- Let $\lambda : E \to \mathbb{R}_+$ be an edge labeling of G.
  A map $\rho : V \to \mathbb{R}^2$ is a *realization* of G compatible with λ iff

$$\|\rho(u) - \rho(v)\| = \lambda(uv)$$

  for all edges uv in E.
- Two realizations ρ_1 and ρ_2 are equivalent iff there exists a direct Euclidean isometry σ of $\mathbb{R}^2$ such that $\rho_1 = \sigma \circ \rho_2$.

Rigid and flexible labelings

A labeling λ of G is:

- *rigid* if the number of realizations of G compatible with λ up to equivalence is positive and finite,
- *flexible* if the number of realizations of G compatible with λ up to equivalence is infinite.

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Let $\bar{u}\bar{v}$ be an edge of G and $C \subset (\mathbb{R} \times \mathbb{R})^{|V|}$ be the zero set of

$$(x_{\bar{u}}, y_{\bar{u}}) = (0, 0)$$
$$(x_{\bar{v}}, y_{\bar{v}}) = (\lambda(\bar{u}\bar{v}), 0)$$
$$(x_u - x_v)^2 + (y_u - y_v)^2 = \lambda(uv)^2, \quad \forall\, uv \in E\,.$$

A labeling λ of G is:

- *rigid* if $0 < |C| < \infty$,
- *flexible* if $|C| = \infty$.

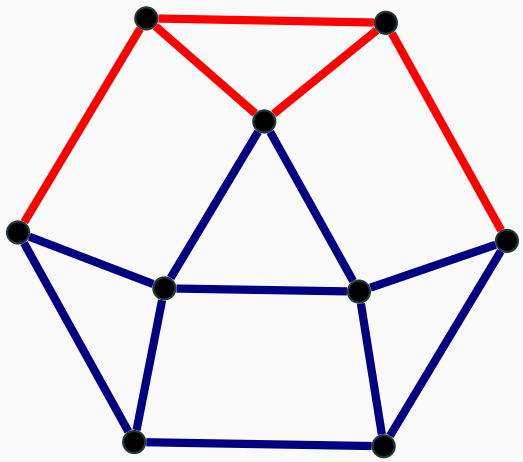
NAC-colorings

Definition

Let $\delta : E \to \{\text{blue, red}\}$ be a coloring of edges.

A cycle in G is an *almost red cycle*, if exactly one of its edges is blue.

A coloring δ is called a *NAC-coloring*, if it is surjective and there are no almost blue or almost red cycles in G.



Combinatorial characterization

Theorem

A connected graph with at least one edge has a flexible labeling if and only if it has a NAC-coloring.

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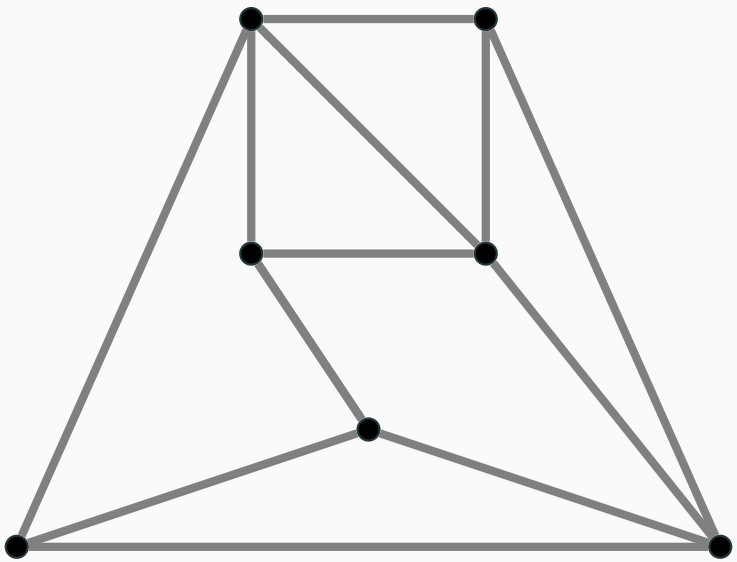

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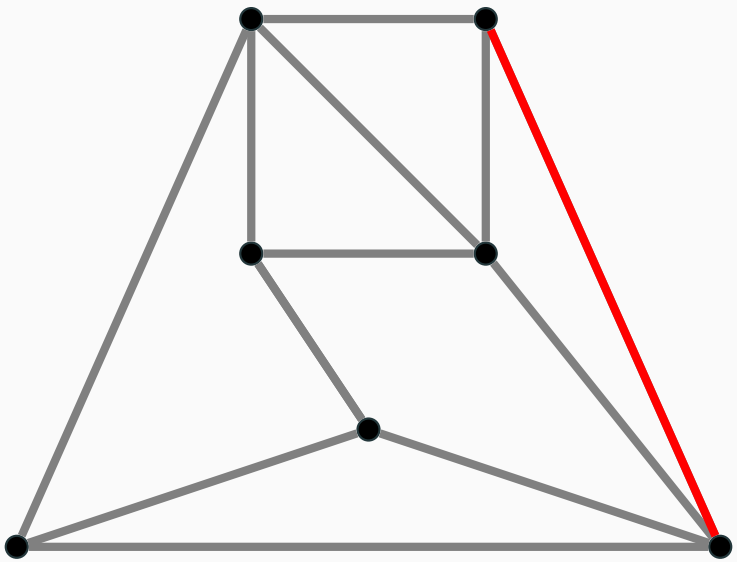

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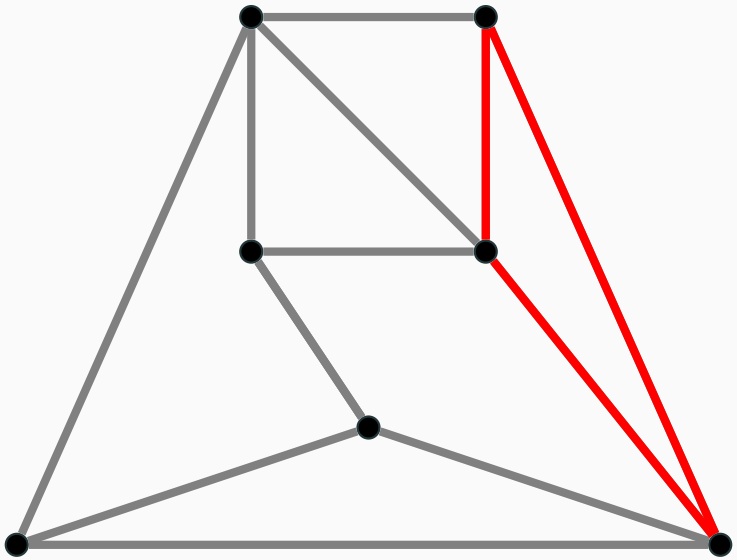

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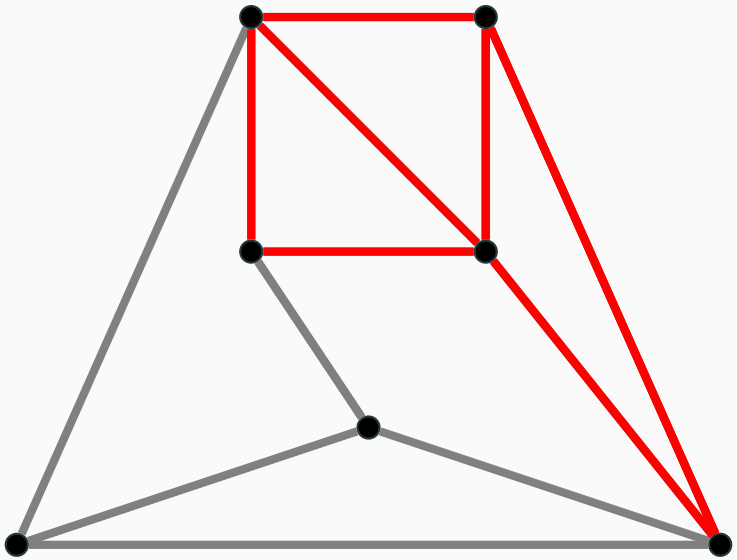

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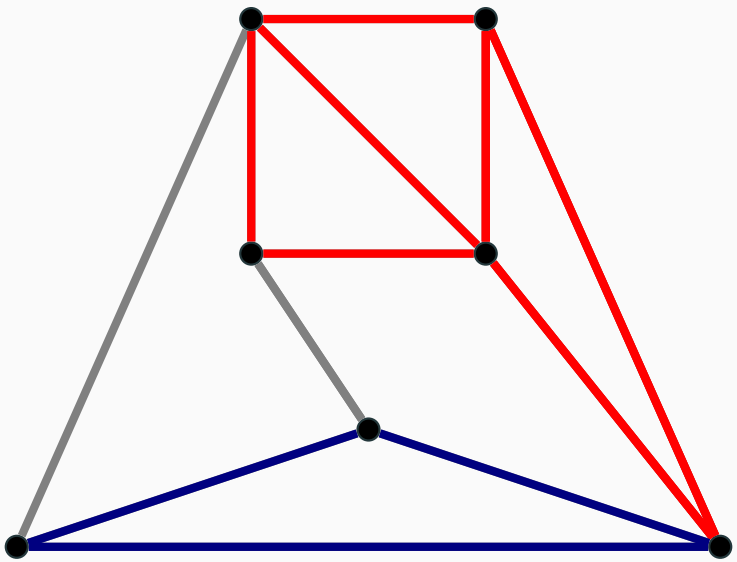

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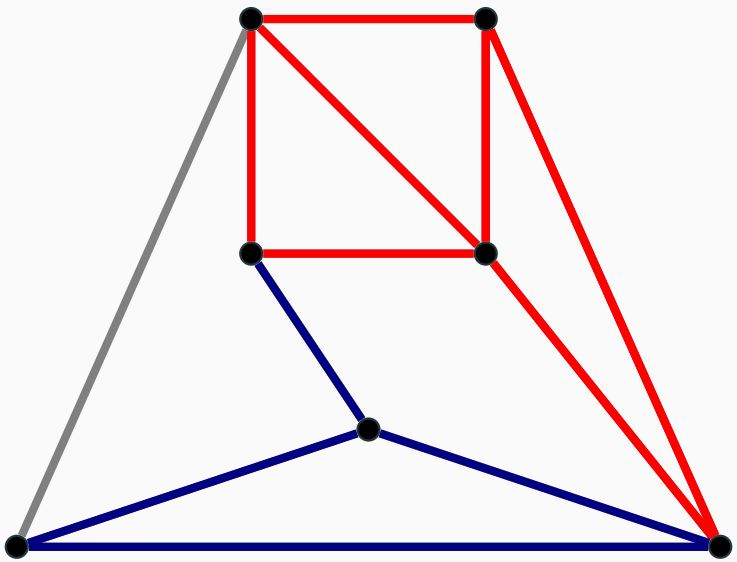

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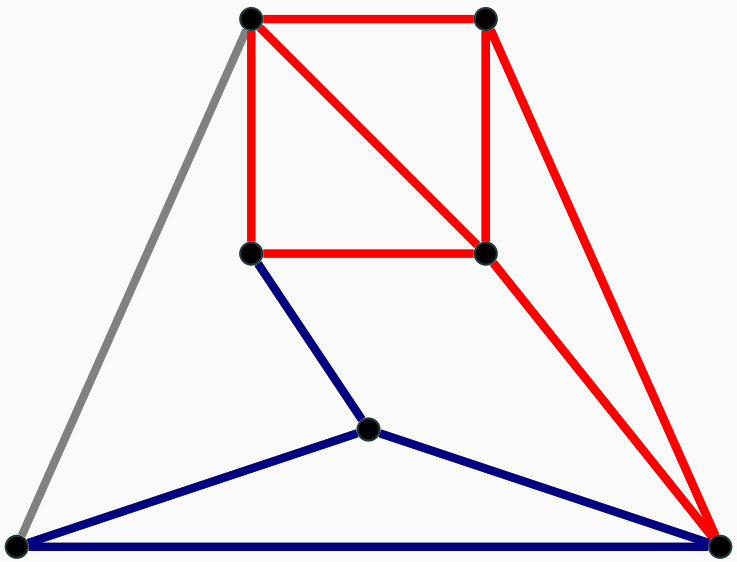
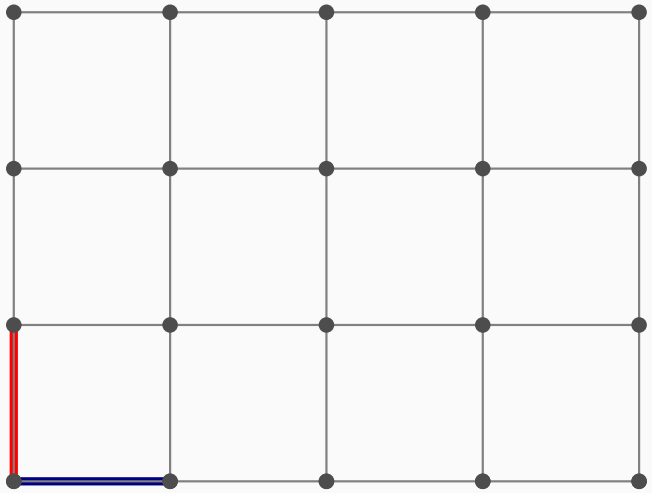
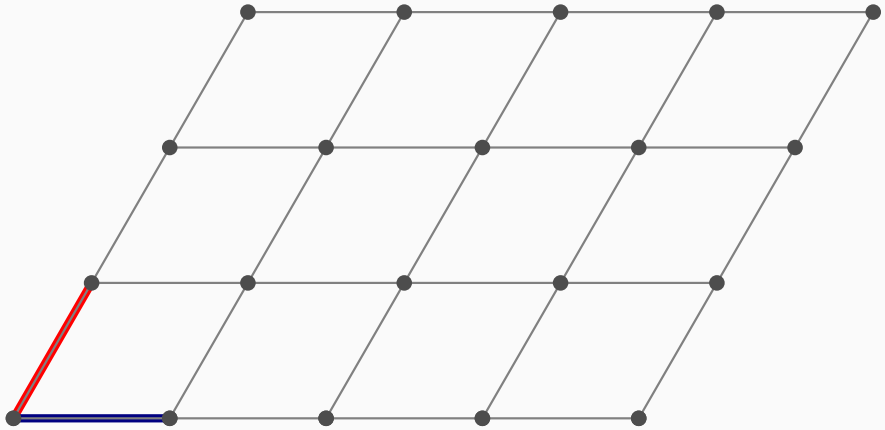


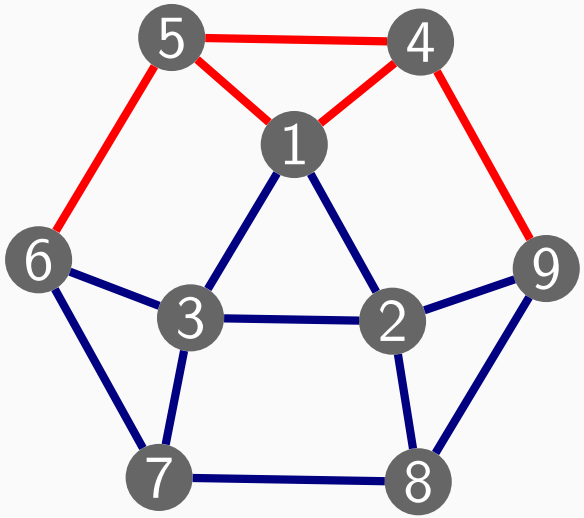
$\Longrightarrow$ no flexible labeling

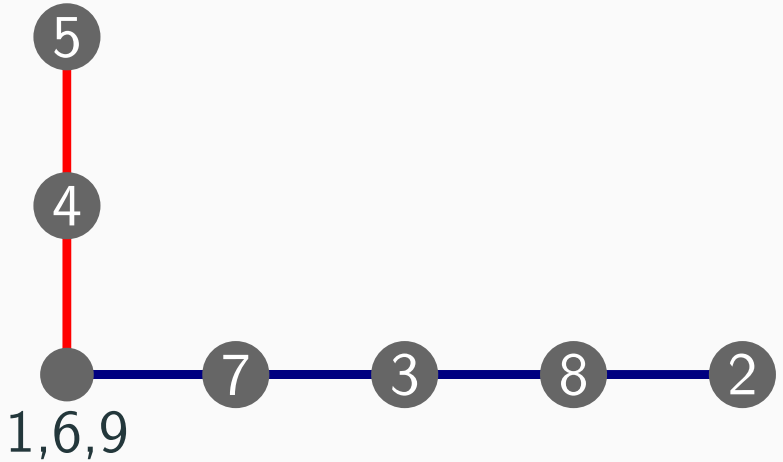
Grid construction

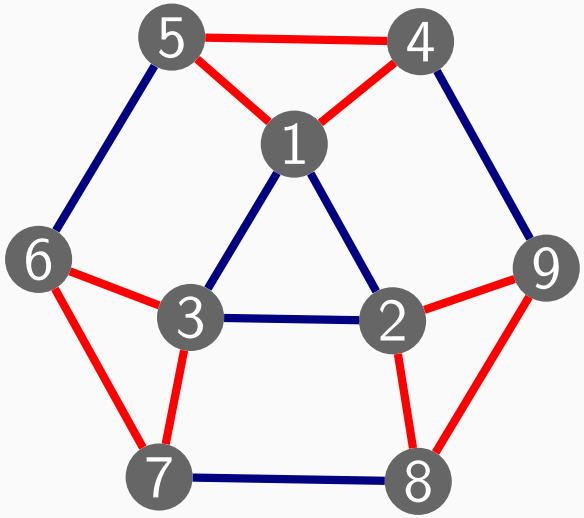


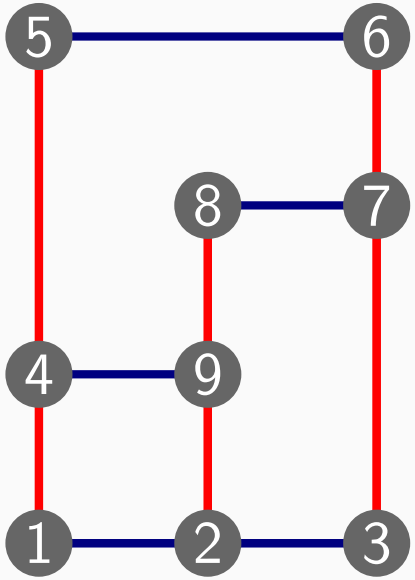


Example

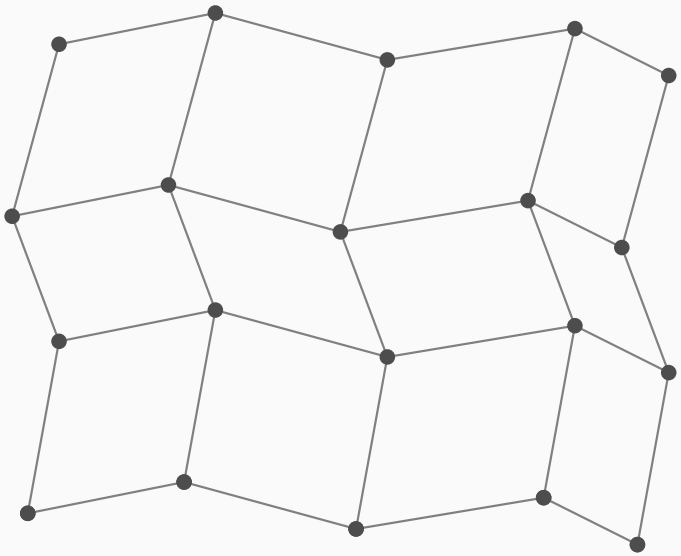


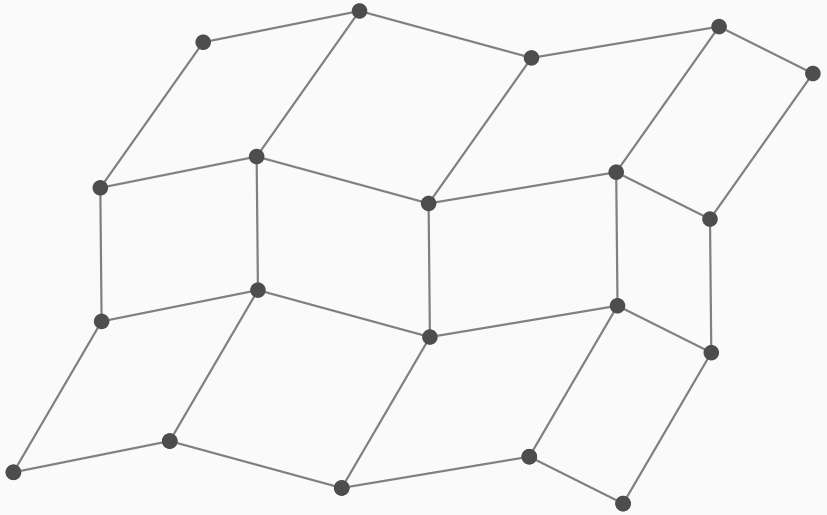






Grid construction II





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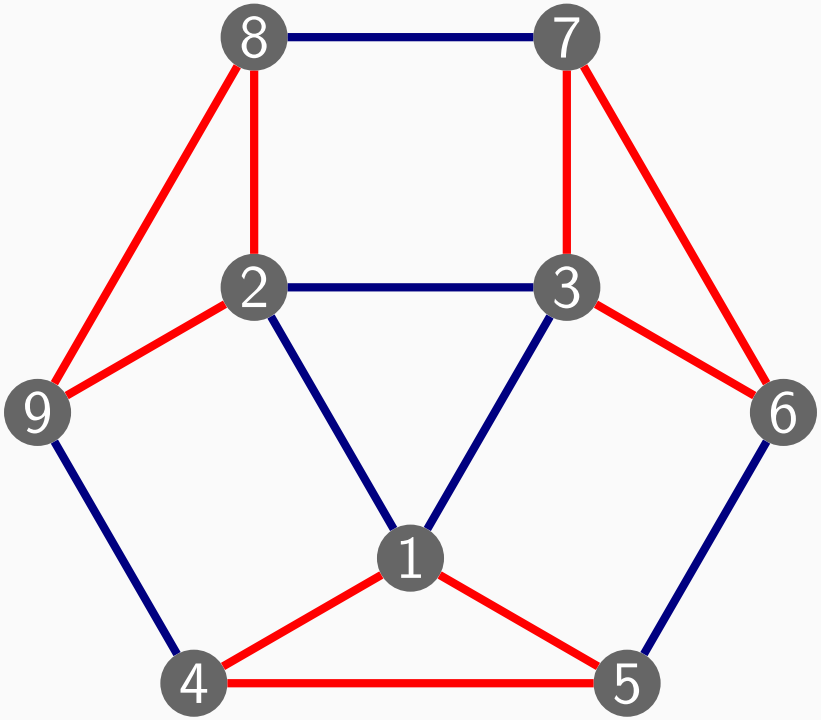

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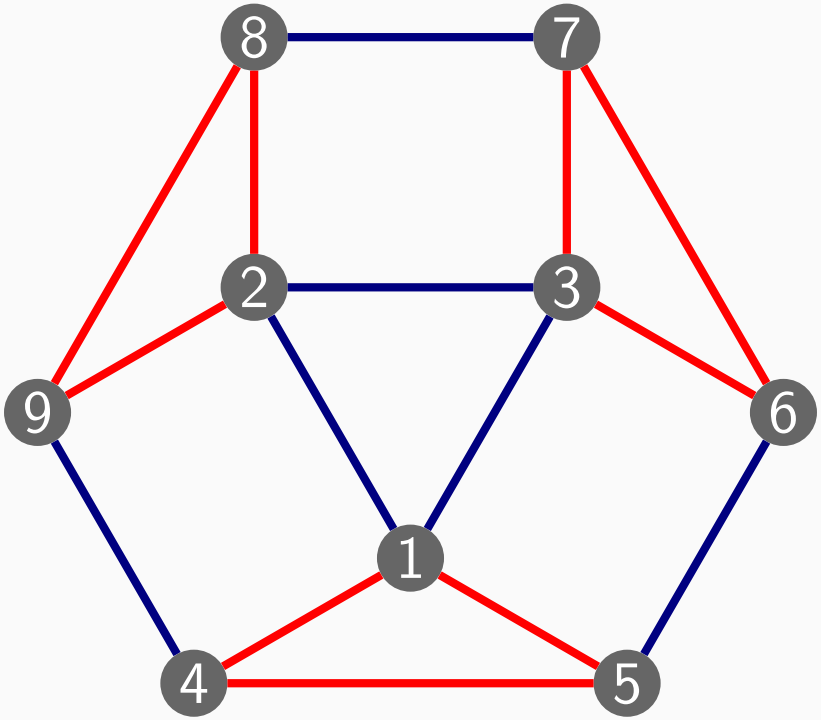

Graphs with NAC-coloring

Definition

Let $\sim'_{\triangle}$ be a relation on $E \times E$ such that $e_1 \sim'_{\triangle} e_2$ iff there exists a triangle subgraph C_3 of G such that $e_1, e_2 \in E_{C_3}$. Let $\sim_{\triangle}$ be the reflexive-transitive closure of $\sim'_{\triangle}$. G is called $\triangle$-*connected* if $e_1 \sim_{\triangle} e_2$ for all $e_1, e_2 \in E$.

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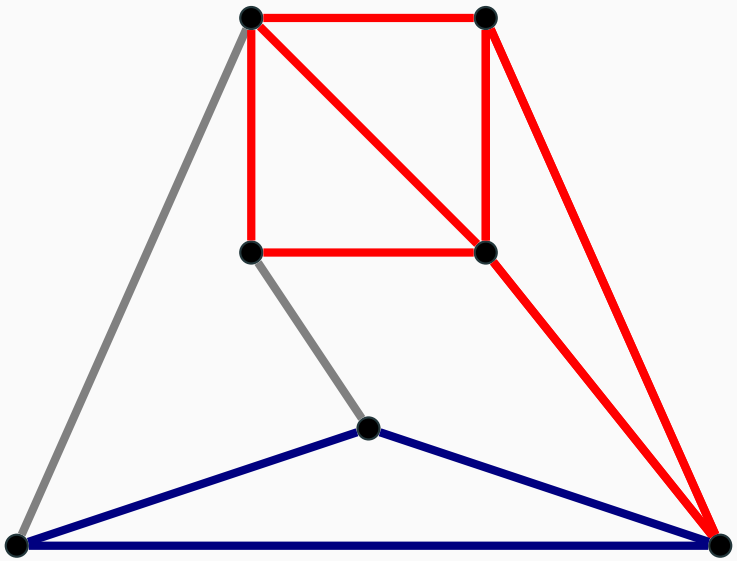

Necessary and sufficient conditions

Theorem

If a graph G has a NAC-coloring, then there is no spanning subgraph H of G such that H is $\triangle$-connected.

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Lemma

If there exists an independent set of vertices V_c which separates G, then G has a NAC-coloring.

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Let E_c be a set of edges of G that are not in any triangle and let E_c separate G. If the subgraph induced by E_c contains no path of length four, then G has a NAC-coloring.

Necessary and sufficient conditions

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If a Laman graph G has a NAC-coloring, then G is not $\triangle$-connected.

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Laman graphs

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A graph $G = (V, E)$ is called *Laman*, if $|E| = 2|V| - 3$ and $|E'| \leq 2|V'| - 3$ for every subgraph (V', E') of G.

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For which Laman graphs is the set of flexible labelings nonempty?

How does the set look like?

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If a Laman graph G has a NAC-coloring, then G is not $\triangle$-connected.

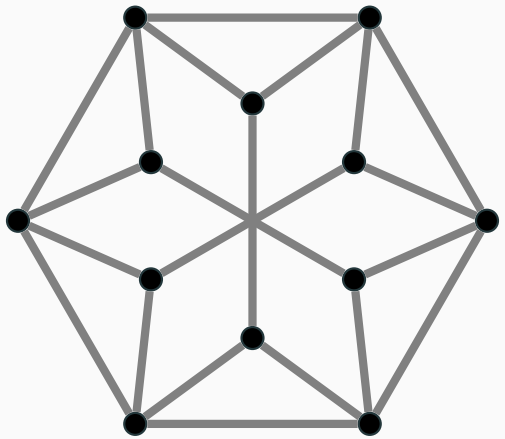
Can we prove the opposite implication?

Problematic graphs

Definition

A Laman graph G is called *problematic*, if the following hold:

1. $\deg(v) \geq 3$ for all $v \in V_G$,
2. if $\deg(v) = 3$, then exactly two neighbours of v are connected by an edge and both have degree at least 4,
3. all vertices are in some triangle $C_3 \subset G$.



Conjecture

Theorem

If a Laman graph G is not $\triangle$-connected, then it has a NAC-coloring, or there exists a problematic graph G' with no NAC-coloring such that G can be constructed from G' by Henneberg steps.

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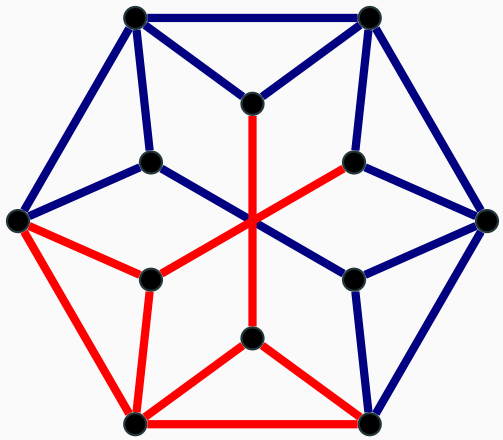
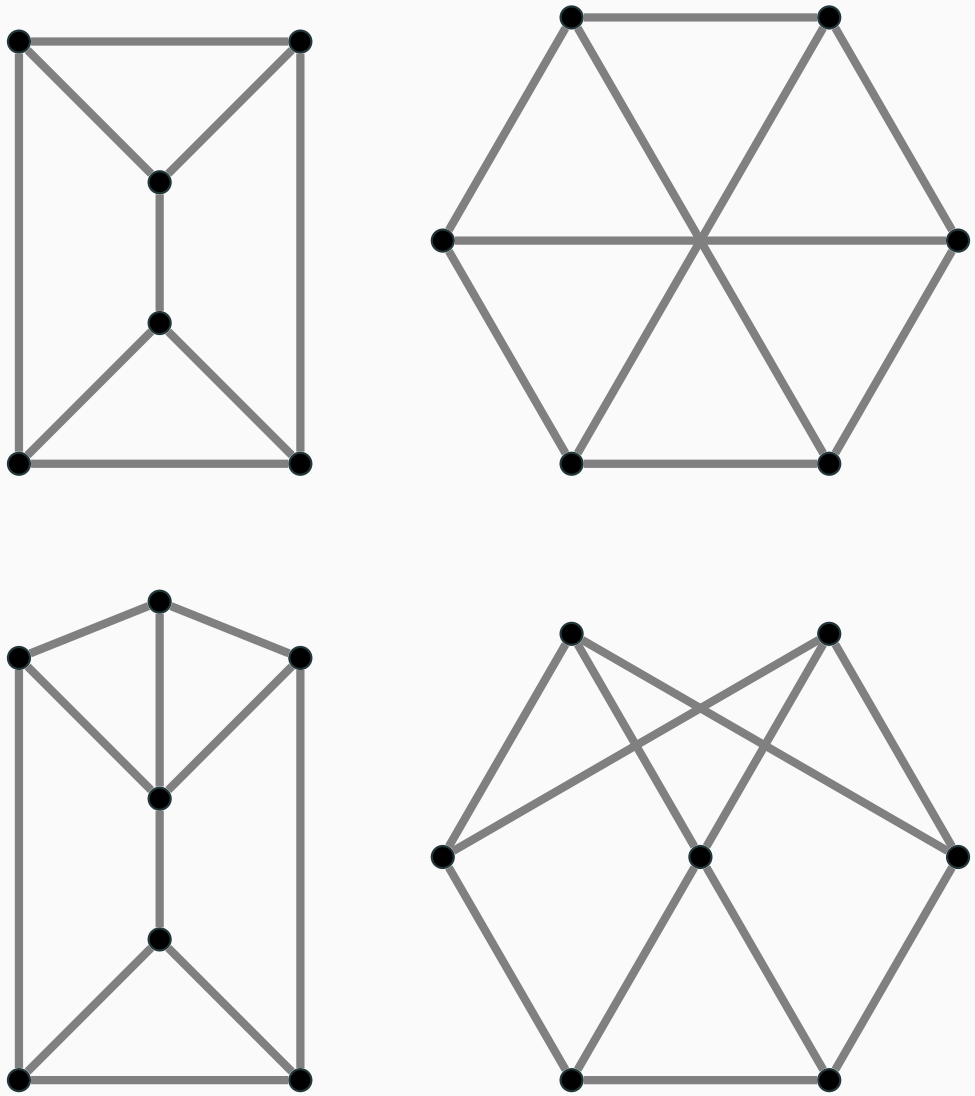
Conjecture

Every problematic Laman graph has a NAC-coloring.



Flexible labelings with infinitely many injective realizations



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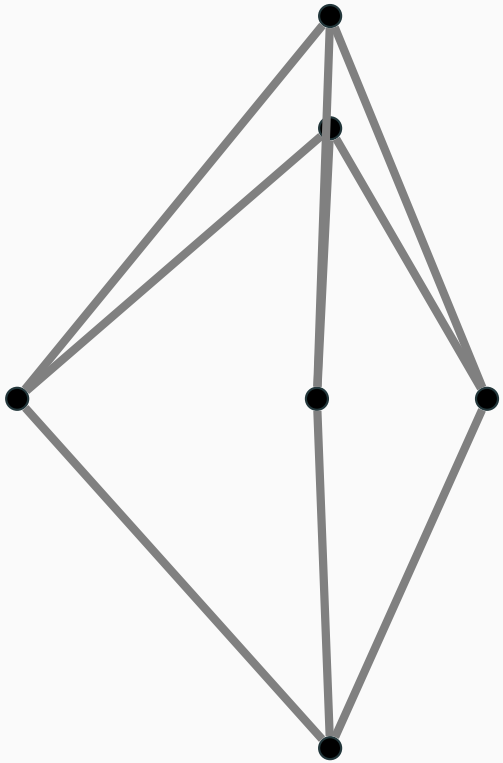

Open questions

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- Sufficient (combinatorial) conditions for existence of a flexible labeling with infinitely many injective realizations?
- Full characterization of all flexible labelings of a Laman graph?

Thank you

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