Rigid Graphs that are Movable

Georg Grasegger, Jan Legerský, Josef Schicho

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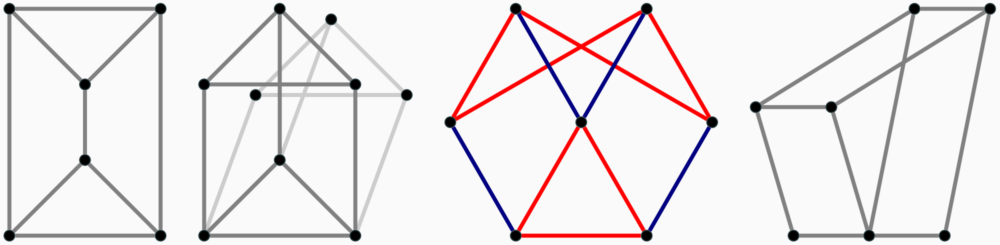

Movable graphs

An edge labeling $\lambda : E \to \mathbb{R}_+$ of a graph $G = (V, E)$ is called *flexible* if there are infinitely many non-congruent realizations $\rho : V \to \mathbb{R}^2$ such that $\|\rho(u) - \rho(v)\| = \lambda(uv)$ for all edges uv in E.

$$
\begin{aligned}
(x_{\bar{u}}, y_{\bar{u}}) &= (0, 0) \\
(x_{\bar{v}}, y_{\bar{v}}) &= (\lambda(\bar{u}\bar{v}), 0) \\
(x_u - x_v)^2 + (y_u - y_v)^2 &= \lambda(uv)^2, \quad \forall\, uv \in E
\end{aligned}
$$

An irreducible component of the solution set is called an *algebraic motion*.

Movable graphs

An edge labeling $\lambda : E \to \mathbb{R}_+$ of a graph $G = (V, E)$ is called *proper flexible* if there are infinitely many non-congruent injective realizations $\rho : V \to \mathbb{R}^2$ such that $\|\rho(u) - \rho(v)\| = \lambda(uv)$ for all edges uv in E.

A graph is called *movable* if it has a proper flexible labeling.

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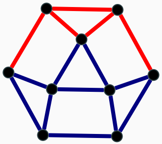
(x_u - x_v)^2 + (y_u - y_v)^2 &= \lambda(uv)^2, \quad \forall\, uv \in E \\
(x_u - x_v)^2 + (y_u - y_v)^2 &\neq 0, \quad \forall\, uv \notin E\,.
\end{aligned}$$

NAC-colorings

Definition

A coloring of edges $\delta : E \rightarrow \{\text{blue, red}\}$ is called a *NAC-coloring*, if it is surjective and for every cycle in G, either all edges in the cycle have the same color, or there are at least two blue and two red edges in the cycle.



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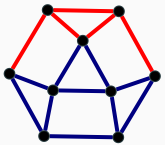
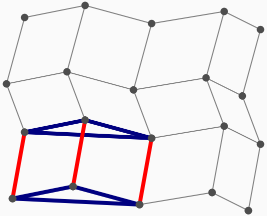


Theorem (GLS)

A connected graph with at least one edge has a flexible labeling if and only if it has a NAC-coloring.

Grid construction



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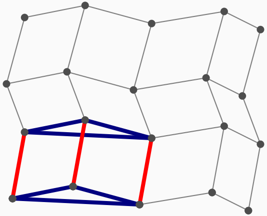
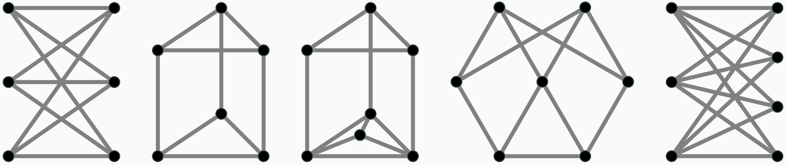


Movable graphs up to 7 vertices

Theorem (GLS)

The maximal movable graphs with at most 7 vertices that are spanned by a Laman graph and have no vertex of degree two are the following:



$K_{3,3}$ L_1 L_2 Q_1 $K_{3,4}$

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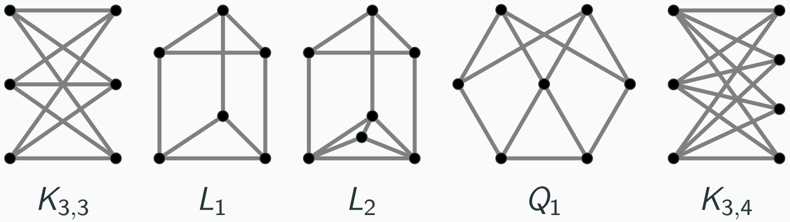
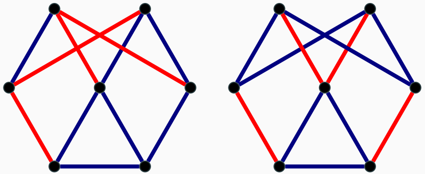


Classification of proper flexible labelings of bipartite graphs is known (Dixon, Maehara, Walter-Husty)

Embedding in $\mathbb{R}^3$

Lemma (GLS)

If there exists an injective realization of G in $\mathbb{R}^3$ such that every edge is parallel to one of the four vectors $(1,0,0),(0,1,0),(0,0,1),(-1,-1,-1)$*, then G is movable.*



Motion of $Q1$

Active NAC-colorings

$$(x_u - x_v)^2 + (y_u - y_v)^2 = \underbrace{((x_v - x_u) + i(y_v - y_u))}_{W_{u,v}} \underbrace{((x_v - x_u) - i(y_v - y_u))}_{Z_{u,v}} = \lambda_{uv}^2$$

$$\sum_{i=0}^{n} W_{u_i,u_{i+1}} = 0 \text{ and } \sum_{i=0}^{n} Z_{u_i,u_{i+1}} = 0 \text{ for every cycle } (u_0, u_1, \ldots, u_n, u_{n+1} = u_0)$$

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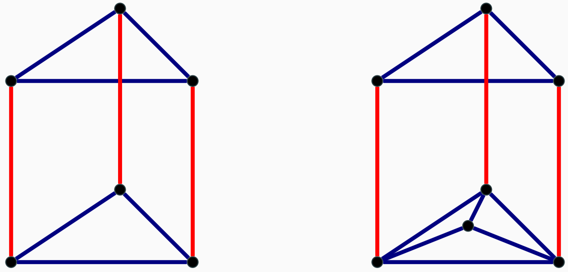
Definition

A NAC-coloring δ of G is called *active* w.r.t. an algebraic motion $\mathcal{C}$ if there exists a valuation ν of the function field of $\mathcal{C}$ and $\alpha \in \mathbb{Q}$ such that $\delta(uv) = \text{red}$ if and only if $\nu(W_{u,v}) > \alpha$ for all $uv \in E$.

Active NAC-colorings of quadrilaterals

Quadrilateral	Motion	active NAC-colorings
Rhombus	parallel	
	degenerate #1 resp. #2	 resp. 
Parallelogram	parallel	
	antiparallel	
Deltoid	nondegenerate	
	degenerate	
General		

Three-prism



Leading coefficient system

A Laurent series parametrization of an algebraic motion gives a valuation

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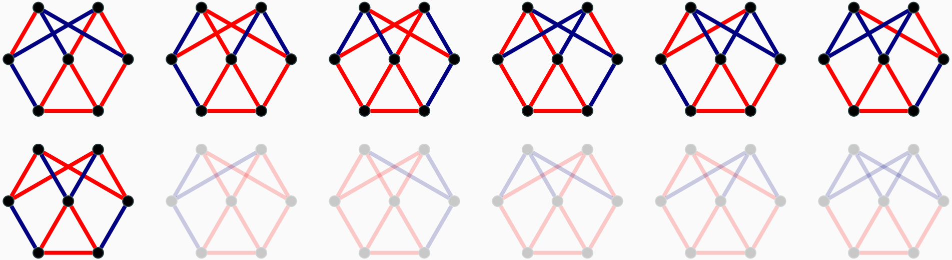
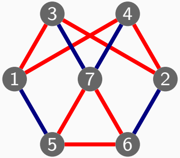


Triangle in $Q1$



$$\implies \lambda_{57}^2 r^2 + \lambda_{67}^2 s^2 + \left(\lambda_{56}^2 - \lambda_{57}^2 - \lambda_{67}^2\right) rs = 0\,, \quad r = \lambda_{24}^2 - \lambda_{23}^2,\ s = \lambda_{14}^2 - \lambda_{13}^2$$

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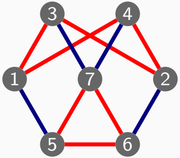

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Considering the equation as a polynomial in r, the discriminant is

$$(\lambda_{56} + \lambda_{57} + \lambda_{67})(\lambda_{56} + \lambda_{57} - \lambda_{67})(\lambda_{56} - \lambda_{57} + \lambda_{67})(\lambda_{56} - \lambda_{57} - \lambda_{67})s^2$$

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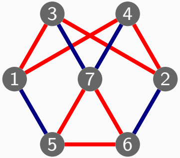

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In combination with the restriction to 4-cycles:

Theorem (GLS)

The vertices 5, 6 and 7 are collinear for every proper flexible labeling of Q_1.

Thank you

jan.legersky@risc.jku.at

jan.legersky.cz